BRST SYMMETRY

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1. Introduction

After the experimental observation of neutral-currents processes in 1973, the requirement for a proof of renormalizability of non-abelian gauge theories predicting the existence of such processes became an essential point in quantum field theory. The discovery [1, 2] of BRST symmetry for the Yang - Mills action made the electroweak model predicting these processes a consistent theoretical framework. As a matter of fact, this underlying symmetry of the gauge fixed action allowed to show it to posses [3, 4] all the required term to make it renormalizable.
BRST symmetry [3, 5] showed to apply to a really wide class of systems of physical interest, and can be easily generalized to a generic system which possessing some basic "gauge symmetry". To this end, we recall that in the literature there exist different formulations for the BRST formalism, with substantial differences from each other. On one side there exists a formulation of BRST symmetry for constrained systems based on canonical quantization methods which is widely diffused [5, 6] and on the other hand there is another approach [3] to derive BRST symmetry based entirely on path integral methods and is applicable to systems with infinite degrees of freedom avoiding those inconsistencies proper of canonical quantization methods we discussed above. In this paper we will follow the latter derivation.

2. DeWitt - Faddev - Popov method

In order to discuss BRST symmetry in the general case [3], let us consider a physical system whose dynamical variables are the "fields" $\{\phi^r\}_r$ and whose action is $I[\phi]$. Let us assume the action and the field integration measure

$$[d\phi] \equiv \prod_r d\phi^r \tag{1}$$

to be invariant with respect to some infinitesimal transformation we will write as

$$\begin{aligned} \phi^r &\rightarrow \phi^r + \epsilon^A \delta_A \phi^r \\ &\equiv \phi^r_\epsilon, \end{aligned} \tag{2}$$

where the indexes r, A are meant to assume both discrete and continuous values. The symmetry (2) will make possible, to derive an expression for the functional integral of

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some operator invariant under (2) which will be useful in the following. As a matter of fact, we consider

$$\mathscr{I} \equiv \int [d\phi]\, \mathscr{G}[\phi]\, B[f[\phi]] \det(\mathscr{F}[\phi])$$

where $\mathscr{G}[\phi]$ is a functional of the fields $\{\phi^r\}_r$ invariant under (2), $\{f_A[\phi]\}_A$ are some functionals of the fields $\{\phi^r\}_r$ which are *not* invariant under (2), $B[f[\phi]]$ is a functional of the $\{f_A[\phi]\}_A$ and

$$\mathscr{F}_{AB}[\phi] \equiv \left.\frac{\delta f_A[\phi_\epsilon]}{\delta \epsilon^B}\right|_{\epsilon=0}$$

and we show the following

Theorem 1. *$\mathscr{I}$ is independent on the choice of the functionals $\{f_A[\phi]\}_A$ and depends on $B[f[\phi]]$ only through a multiplicative field-independent constant.*

We can use Theorem 1 to gain the expression for the functional integral of an operator invariant under (2) we were searching for. Let us consider a functional $V[\phi]$ invariant under (2) and write the variation of $\{f_A[\phi]\}_A$ under (2) as

$$\begin{aligned} f_A[\phi] &\rightarrow f_A[\phi] + \epsilon^B \delta_B f_A[\phi] \\ &= f_A[\phi] + \epsilon^B \left.\frac{\delta f_A[\phi_\epsilon]}{\delta \epsilon^B}\right|_{\epsilon=0}, \end{aligned}$$

setting $\Omega \equiv \int \prod_A df_A$ and taking

$$\mathscr{G}[\phi] = e^{iI[\phi]} V[\phi].$$

Using Theorem 1 we can thus derive this important identity originally derived by a B. S. DeWitt, L. D. Fadeev e V. N. Popov [7, 8]

$$\begin{aligned} \mathscr{I} &= \int [d\phi]\, e^{iI[\phi]} V[\phi]\, B[f[\phi]] \det(\delta_B f_A[\phi]) \qquad (3) \\ &= \frac{C}{C|_{B=1}} \mathscr{I}|_{B=1} \\ &= \frac{C}{\Omega} \int [d\phi]\, e^{iI[\phi]} V[\phi]. \end{aligned}$$

3. BRST transformation

In order to introduce the BRST transformation we write the functional $B[f]$ by means of its Fourier transform

$$B[f] = \int [dh] e^{ih^A f_A} \mathscr{B}[h] \qquad (4)$$

where $[dh] \equiv \prod_A dh^A$. We recall [9] the the determinant of the matrix $\delta_A f_B[\phi]$ can be written by means of a functional integral on some Grassmann variables $\left\{c^A, c^{*A}\right\}_A$, called *ghost* ed *antighost fields* respectively[1]

$$\det(\delta_A f_B[\phi]) \propto \int [dc^*]\, [dc]\, e^{ic^{*B} c^A \delta_A f_B[\phi]} \qquad (5)$$

[1]We stress that c^A and c^{*A}, being completely independent on each other, are *not* related by complex conjugation.

where $[dc] \equiv \prod_A dc^A,\ [dc^*] \equiv \prod_A dc^{*A}$. Substituting (4), (5) in (3) we have

$$\begin{aligned}
\mathscr{I} &\propto \int [d\phi]\, e^{iI[\phi]} V[\phi] \int [dh] e^{ih^A f_A[\phi]} \mathscr{B}[h] \int [dc^*]\,[dc]\, e^{ic^{*B} c^A \delta_A f_B[\phi]} \qquad (6) \\
&= \int [d\phi]\,[dh]\,[dc^*]\,[dc]\, e^{iI[\phi]} e^{ih^A f_A[\phi]} e^{ic^{*B} c^A \delta_A f_B[\phi]} V[\phi]\, \mathscr{B}[h] \\
&= \int [d\phi]\,[dh]\,[dc^*]\,[dc]\, e^{iI_{NEW}[\phi,h,c,c^*]} V[\phi]\, \mathscr{B}[h]
\end{aligned}$$

with

$$I_{NEW}[\phi,h,c,c^*] \equiv I[\phi] + h^A f_A[\phi] + c^{*B} c^A \delta_A f_B[\phi]. \qquad (7)$$

Now we set $\{\psi_i\}_i \equiv \left\{ \{\phi^r\}_r, \left\{h^A\right\}_A, \left\{c^A, c^{*A}\right\}_A \right\}$ and give the following

Definition 2. Given a functional $F[\psi]$ se $\delta F[\psi] = (\delta\psi_i)\, G^i[\psi]$ we set $G^i[\psi] \equiv \frac{\delta_L F[\psi]}{\delta \psi_i}$.

Definition 3. Given a functional $F[\psi]$, we will write the first order variation of $F[\psi]$ under a transformation (2) as $\delta F[\psi] = \epsilon^A \delta_A F[\psi]$.

Definition 4. The structure constants $f^C{}_{AB}$ are given by

$$[\delta_B, \delta_C] = f^A{}_{BC} \delta_A. \qquad (8)$$

Definition 5. The *Slavnov's operator* is given by

$$s \equiv c^A (\delta_A \phi^r) \frac{\delta_L}{\delta \phi^r} - \frac{1}{2} c^B c^C f^A{}_{BC} \frac{\delta_L}{\delta c^A} - h^A \frac{\delta_L}{\delta c^{*A}}. \qquad (9)$$

Remark 6. Applying the Definitions 2, 3 to a generic functional $F[\phi]$ we get

$$\begin{aligned}
\delta F[\phi] &= (\delta\psi_i) \frac{\delta_L F[\phi]}{\delta \psi_i} \\
&= (\delta\phi^s) \frac{\delta_L F[\phi]}{\delta \phi^s}
\end{aligned}$$

and taking $\delta\phi^s = \epsilon^A \delta_A \phi^s$

$$\begin{aligned}
\delta F[\phi] &= \epsilon^A \delta_A F[\phi] \\
&= (\delta\phi^s) \frac{\delta_L F[\phi]}{\delta \phi^s} \\
&= \epsilon^A (\delta_A \phi^s) \frac{\delta_L F[\phi]}{\delta \phi^s}
\end{aligned}$$

so, being $\left\{\epsilon^A\right\}_A$ completely arbitrary, we find

$$\delta_A F[\phi] = \frac{\delta_L F[\phi]}{\delta \phi^s} \delta_A \phi^s. \qquad (10)$$

We are now in the position to show that the action (7) even if, containing the functionals $\{f_A[\phi]\}_A$, is not invariant under (2), is symmetric with respect to another transformation, originally discovered by C. Becchi, A. Rouet, R. Stora e I. V. Tyutin within gauge theories [3, 5] and known as *BRST transformation*, acting on a generic functional $F[\psi]$ in the following way

$$F[\psi] \to F[\psi] + \theta s F[\psi] \qquad (11)$$

where θ is an "infinitesimal" Grassmann variable .

We observe that the transformation (11), acting on a functional $F[\phi]$, is just a transformation (2) with infinitesimal parameters $\epsilon^A = \theta c^A$. In fact under (11)

$$\begin{aligned} F[\phi] \quad & \to F[\phi] + \theta s F[\phi] \\ & = F[\phi] + \theta c^A (\delta_A \phi^r) \tfrac{\delta_L F[\phi]}{\delta \phi^r} \\ & = F[\phi] + \left(\theta c^A\right) \delta_A F[\phi] \end{aligned} \tag{12}$$

which, by means of 3,is just a transformation (2) with infinitesimal parameter $\epsilon^A = \theta c^A$.

To show that (11) is a symmetry for $I_{NEW}[\phi, h, c, c^*]$ we first need to show that the BRST transformation is nilpotent i.e. $s^2 = 0$. Using (9) we have

$$\begin{aligned} s^2 \ = \ & \left[c^A (\delta_A \phi^r) \frac{\delta_L}{\delta \phi^r} - \frac{1}{2} c^B c^C f^A{}_{BC} \frac{\delta_L}{\delta c^A} - h^A \frac{\delta_L}{\delta c^{*A}}\right]\left[c^D (\delta_D \phi^s) \frac{\delta_L}{\delta \phi^s} - \frac{1}{2} c^D c^E f^F{}_{DE} \times\right. \\ & \left. \times \frac{\delta_L}{\delta c^F} - h^D \frac{\delta_L}{\delta c^{*D}}\right] = \\ = \ & c^A (\delta_A \phi^r) \left\{ \frac{\delta_L \left(c^D \delta_D \phi^s\right)}{\delta \phi^r} \frac{\delta_L}{\delta \phi^s} + c^D (\delta_D \phi^s) \frac{\delta_L}{\delta \phi^r} \frac{\delta_L}{\delta \phi^s} - \frac{1}{2}\left[\frac{\delta_L \left(c^D c^E f^F{}_{DE}\right)}{\delta \phi^r} \frac{\delta_L}{\delta c^F} + c^D c^E \times\right.\right. \\ & \left.\left. \times f^F{}_{DE} \frac{\delta_L}{\delta \phi^r} \frac{\delta_L}{\delta c^F}\right] - h^D \frac{\delta_L}{\delta \phi^r} \frac{\delta_L}{\delta c^{*D}} \right\} - \frac{1}{2} c^B c^C f^A{}_{BC} \left\{ \frac{\delta_L \left(c^D \delta_D \phi^s\right)}{\delta c^A} \frac{\delta_L}{\delta \phi^s} - c^D (\delta_D \phi^s) \times\right. \\ & \left. \times \frac{\delta_L}{\delta c^A} \frac{\delta_L}{\delta \phi^s} - \frac{1}{2}\left[\frac{\delta_L \left(c^D c^E f^F{}_{DE}\right)}{\delta c^A} \frac{\delta_L}{\delta c^F} + c^D c^E f^F{}_{DE} \frac{\delta_L}{\delta c^A} \frac{\delta_L}{\delta c^F}\right] - h^D \frac{\delta_L}{\delta c^A} \frac{\delta_L}{\delta c^{*D}} \right\} + \\ & -h^A \left[- c^D (\delta_D \phi^s) \frac{\delta_L}{\delta c^{*A}} \frac{\delta_L}{\delta \phi^s} - \frac{1}{2} c^D c^E f^F{}_{DE} \frac{\delta_L}{\delta c^{*A}} \frac{\delta_L}{\delta c^F} - h^D \frac{\delta_L}{\delta c^{*A}} \frac{\delta_L}{\delta c^{*D}} \right] = \\ = \ & c^A (\delta_A \phi^r) \left[c^D \frac{\delta_L (\delta_D \phi^s)}{\delta \phi^r} \frac{\delta_L}{\delta \phi^s} - \frac{1}{2}\left(c^D c^E \frac{\delta_L f^F{}_{DE}}{\delta \phi^r} \frac{\delta_L}{\delta c^F} + c^D c^E f^F{}_{DE} \frac{\delta_L}{\delta \phi^r} \frac{\delta_L}{\delta c^F}\right) + \right. \\ & \left. -h^D \frac{\delta_L}{\delta \phi^r} \frac{\delta_L}{\delta c^{*D}} \right] - \frac{1}{2} c^B c^C f^A{}_{BC} \left[(\delta_A \phi^s) \frac{\delta_L}{\delta \phi^s} - c^D (\delta_D \phi^s) \frac{\delta_L}{\delta c^A} \frac{\delta_L}{\delta \phi^s} - f^F{}_{AD} c^D \frac{\delta_L}{\delta c^F} + \right. \\ & \left. -h^D \frac{\delta_L}{\delta c^A} \frac{\delta_L}{\delta c^{*D}} \right] + h^A \left[c^D (\delta_D \phi^s) \frac{\delta_L}{\delta c^{*A}} \frac{\delta_L}{\delta \phi^s} + \frac{1}{2} c^D c^E f^F{}_{DE} \frac{\delta_L}{\delta c^{*A}} \frac{\delta_L}{\delta c^F} \right] = \end{aligned} \tag{13}$$

$$
\begin{aligned}
&= \frac{1}{2}c^A c^D \left[\left(\delta_{[A}\phi^r\right) \frac{\delta_L\left(\delta_{D]}\phi^s\right)}{\delta\phi^r} - f^C{}_{AD}\delta_C\phi^s \right] \frac{\delta_L}{\delta\phi^s} + \frac{1}{2}c^B c^C c^D \left[f^A{}_{BC} f^F{}_{AD} + \right. \qquad (14)\\
&\quad \left. - \left(\delta_D\phi^r\right) \frac{\delta_L f^F{}_{BC}}{\delta\phi^r} \right] \frac{\delta_L}{\delta c^F} - \frac{1}{2}c^A c^D c^E \left(\delta_A\phi^r\right) f^F{}_{DE} \frac{\delta_L}{\delta\phi^r}\frac{\delta_L}{\delta c^F} + \frac{1}{2}c^D c^B c^C f^A{}_{BC}\left(\delta_D\phi^s\right) \times \\
&\quad \times \frac{\delta_L}{\delta\phi^s}\frac{\delta_L}{\delta c^A} + \frac{1}{2}c^D c^E h^A f^F{}_{DE}\frac{\delta_L}{\delta c^{*A}}\frac{\delta_L}{\delta c^F} - c^A\left(\delta_A\phi^r\right)h^D\frac{\delta_L}{\delta\phi^r}\frac{\delta_L}{\delta c^{*D}} + c^D\left(\delta_D\phi^s\right)h^A\frac{\delta_L}{\delta c^{*A}}\frac{\delta_L}{\delta\phi^s} + \\
&\quad + \frac{1}{2}c^B c^C f^A{}_{BC} h^D \frac{\delta_L}{\delta c^A}\frac{\delta_L}{\delta c^{*D}} = \\
&= \frac{1}{2}c^A c^B \left[\frac{\delta_L\left(\delta_{[B}\phi^r\right)}{\delta\phi^s}\delta_{A]}\phi^s - f^C{}_{AB}\delta_C\phi^r \right] \frac{\delta_L}{\delta\phi^r} - \frac{1}{2}c^B c^C c^D \left(f^E{}_{BC} f^A{}_{DE} + \frac{\delta_L f^A{}_{BC}}{\delta\phi^r}\delta_D\phi^r \right) \times \\
&\quad \times \frac{\delta_L}{\delta c^A}.
\end{aligned}
$$

By means of (13) it's easy to show that the BRST transformation is nilpotent. In order to show this, we consider (10) with $F[\phi] = \delta_B\phi^r$ and we take the antisymmetric part with respect to A, B using (8); finally we get

$$
\begin{aligned}
\delta_{[A}\delta_{B]}\phi^r &= \frac{\delta_L\left(\delta_{[B}\phi^r\right)}{\delta\phi^s}\delta_{A]}\phi^s \\
&= f^C{}_{AB}\delta_C\phi^r
\end{aligned}
$$

showing that the first term in square brackets in the last passage of (13) equals zero. Besides we consider Jacobi's identity

$$
\begin{aligned}
0 &= \left[\left[\delta_A, \delta_B\right], \delta_C\right] + \left[\left[\delta_B, \delta_C\right], \delta_A\right] + \left[\left[\delta_C, \delta_A\right], \delta_B\right] \qquad (15)\\
&= \left[f^D{}_{AB}\delta_D, \delta_C\right] + \left[f^D{}_{BC}\delta_D, \delta_A\right] + \left[f^D{}_{CA}\delta_D, \delta_B\right] \\
&= f^D{}_{AB}\delta_D\delta_C - \delta_C\left(f^D{}_{AB}\delta_D\right) + f^D{}_{BC}\delta_D\delta_A - \delta_A\left(f^D{}_{BC}\delta_D\right) + f^D{}_{CA}\delta_D\delta_B - \delta_B\left(f^D{}_{CA}\delta_D\right) \\
&= f^D{}_{AB}\delta_{[D}\delta_{C]} - \left(\delta_C f^D{}_{AB}\right)\delta_D + f^D{}_{BC}\delta_{[D}\delta_{A]} - \left(\delta_A f^D{}_{BC}\right)\delta_D + f^D{}_{CA}\delta_{[D}\delta_{B]} - \left(\delta_B f^D{}_{CA}\right)\delta_D \\
&= f^D{}_{AB} f^E{}_{DC}\delta_E + f^D{}_{BC} f^E{}_{DA}\delta_E + f^D{}_{CA} f^E{}_{DB}\delta_E - \left(\delta_C f^D{}_{AB}\right)\delta_D - \left(\delta_A f^D{}_{BC}\right)\delta_D + \\
&\quad - \left(\delta_B f^D{}_{CA}\right)\delta_D \\
&= \left(f^D{}_{AB} f^E{}_{DC} + f^D{}_{BC} f^E{}_{DA} + f^D{}_{CA} f^E{}_{DB} - \delta_C f^E{}_{AB} - \delta_A f^E{}_{BC} - \delta_B f^E{}_{CA}\right)\delta_E \\
&= \frac{1}{2}\left[f^D{}_{[AB]} f^E{}_{DC} + f^D{}_{[BC]} f^E{}_{DA} + f^D{}_{[CA]} f^E{}_{DB} - \left(\delta_C f^E{}_{[AB]} + \delta_A f^E{}_{[BC]} + \delta_B f^E{}_{[CA]} \right) \right] \delta_E \\
&= \frac{1}{2}\left(f^D{}_{[AB} f^E{}_{DC]} - \delta_{[C} f^E{}_{AB]} \right)\delta_E.
\end{aligned}
$$

Looking at (8) it's easy to realize that the structure constants $f^C{}_{AB}$ depend only on the fields $\{\phi^r\}_r$. Thus we can employ (10) with $F[\phi] = f^C{}_{AB}$ and gain

$$
\delta_D f^C{}_{AB} = \frac{\delta_L f^C{}_{AB}}{\delta\phi^s}\delta_D\phi^s. \qquad (16)
$$

Substituting (16) in (15) we get

$$\left(f^{D}{}_{[AB}f^{E}{}_{DC]}-\frac{\delta_L f^{E}{}_{[AB}}{\delta\phi^{s}}\delta_{C]}\phi^{s}\right)\delta_E=0$$

which implies

$$f^{D}{}_{[AB}f^{E}{}_{C]D}+\frac{\delta_L f^{E}{}_{[AB}}{\delta\phi^{s}}\delta_{C]}\phi^{s}=0. \tag{17}$$

(17) is just a generalization of the Jacobi's identity for field-dependent structure constants. Using (17) the second term in the last passage of (13) is

$$\begin{aligned}\frac{1}{2}c^{B}c^{C}c^{D}\left(f^{E}{}_{BC}f^{A}{}_{DE}+\frac{\delta_L f^{A}{}_{BC}}{\delta\phi^{r}}\delta_D\phi^{r}\right)\frac{\delta_L}{\delta c^{A}} &= \frac{1}{2}c^{B}c^{C}c^{D}\frac{1}{3!}\left(f^{E}{}_{[BC}f^{A}{}_{D]E}+\frac{\delta_L f^{A}{}_{[BC}}{\delta\phi^{r}}\delta_{D]}\phi^{r}\right)\times\\ &\quad\times\frac{\delta_L}{\delta c^{A}}\\ &= 0,\end{aligned}$$

so

$$s^2=0. \tag{18}$$

To show the action (7) to be invariant under (11) we observe that, using (10), we have

$$\begin{aligned}s\left(c^{*A}f_A[\phi]\right) &= c^{B}\left(\delta_B\phi^{r}\right)\frac{\delta_L\left(c^{*A}f_A[\phi]\right)}{\delta\phi^{r}}-h^{B}\frac{\delta_L\left(c^{*A}f_A[\phi]\right)}{\delta c^{*B}} \\ &= c^{B}\left(\delta_B\phi^{r}\right)c^{*A}\frac{\delta_L f_A[\phi]}{\delta\phi^{r}}-h^{B}\frac{\delta_L c^{*A}}{\delta c^{*B}}f_A[\phi]\\ &= -\left[c^{*B}c^{A}\left(\delta_A\phi^{r}\right)\frac{\delta_L f_B[\phi]}{\delta\phi^{r}}+h^{A}f_A[\phi]\right]\\ &= -\left(c^{*B}c^{A}\delta_A f_B[\phi]+h^{A}f_A[\phi]\right).\end{aligned} \tag{19}$$

Substituting (19) in (7) we get

$$I_{NEW}[\phi,h,c,c^*]=I[\phi]-s\left(c^{*A}f_A[\phi]\right). \tag{20}$$

Thus, using (18) and the fact that the transformation (11), acting on $I[\phi]$, is a transformation (2) and that $I[\phi]$ is invariant under (2), we can show that under (11)

$$\begin{aligned}I_{NEW}[\phi,h,c,c^*] &\rightarrow I_{NEW}[\phi,h,c,c^*]+\theta s I_{NEW}[\phi,h,c,c^*]\\ &= I_{NEW}[\phi,h,c,c^*]+\theta s\left[I[\phi]-s\left(c^{*A}f_A[\phi]\right)\right]\\ &= I_{NEW}[\phi,h,c,c^*]+\theta s I[\phi]\\ &= I_{NEW}[\phi,h,c,c^*].\end{aligned}$$

This shows the invariance of $I_{NEW}[\phi,h,c,c^*]$ under the BRST transformation.

4. BRST charge

In the quantum theory the transformation (11) acts on states in the Hilbert space $\mathcal{H}$. We can thus define a fermionic operator Q, known as *BRST charge*, so that the variation under a transformation (11) of a generic operator $\Phi[\psi]$ is

$$\delta_\theta \Phi[\psi] = -i[\theta Q, \Phi[\psi]]. \tag{21}$$

If $\Phi[\psi]$ is bosonic or fermionic, (21) becomes

$$\begin{aligned} \delta_\theta \Phi[\psi] &= -i(\theta Q \Phi[\psi] - \Phi[\psi]\theta Q) \\ &= -i\theta(Q\Phi[\psi] \mp \Phi[\psi] Q) \\ &= -i\theta[Q, \Phi[\psi]]_\mp . \end{aligned} \tag{22}$$

Employing the fact that the BRST transformation is nilpotent, it is easy to show that $Q^2 = 0$. As a matter of fact, by means of (11), (22) we have $\delta_\theta \Phi[\psi] = \theta s \Phi[\psi] = -i\theta[Q, \Phi[\psi]]_\mp$. Thus, using (18) we get

$$\begin{aligned} 0 &= s^2 \Phi[\psi] \\ &= s\left(-i[Q, \Phi[\psi]]_\mp\right) \\ &= -is[Q, \Phi[\psi]]_\mp \\ &= -\left[Q, [Q, \Phi[\psi]]_\mp\right]_\pm \\ &= -[Q(Q\Phi[\psi] \mp \Phi[\psi] Q) \pm (Q\Psi[\psi] \mp \Phi[\psi] Q) Q] \\ &= -\left(Q^2 \Phi[\psi] - \Phi[\psi] Q^2\right) \\ &= -\left[Q^2, \Phi[\psi]\right] \end{aligned}$$

from which follows that $Q^2 \propto I$ o $Q^2 = 0$. But, having Q a non vanishing ghost number [10], we necessarily have

$$Q^2 = 0.$$

The charge Q has a really important role in selecting physical states in $\mathcal{H}$. To show this, we consider some operators $O_A, O_B, \ldots$ depending just on the fields $\{\phi^r\}_r$ and invariant under (2). *If* we can express the vacuum T-product of an arbitrary number of operators $O_A, O_B, \ldots$ as a sum over the paths [3, 9], we get

$$\langle 0 | T(O_A O_B \cdots) | 0 \rangle = \frac{\int [d\phi]\, e^{iI[\phi]} O_A O_B \cdots}{\int [d\phi]\, e^{iI[\phi]}}$$

which, using (3) and (6), becomes

$$\langle 0 | T(O_A O_B \cdots) | 0 \rangle = \frac{\int [d\phi][dh][dc^*][dc]\, e^{iI_{NEW}[\phi,h,c,c^*]} O_A O_B \cdots \mathscr{B}[h]}{\int [d\phi][dh][dc^*][dc]\, e^{iI_{NEW}[\phi,h,c,c^*]} \mathscr{B}[h]}. \tag{23}$$

Given two physical states $|\alpha\rangle, |\beta\rangle$ we know [9] that the amplitude $\langle \alpha | \beta \rangle$ can be expressed in terms of a path integral. Anyway, looking at (23), (7) we see that $\langle \alpha | \beta \rangle$ depends on the functionals $\{f_A[\phi]\}_A$, whose choice must not change the amplitude between two physical states. Thus we require that, varying $\Psi \equiv c^{*A} f_A [\phi]$, the amplitude $\langle \alpha | \beta \rangle$ remains unchanged. Using (22) we get

$$
\begin{aligned}
\delta \langle \alpha | \beta \rangle &= \langle \alpha | i\delta (s\Psi) | \beta \rangle \\
&= i \langle \alpha | s (\delta\Psi) | \beta \rangle \\
&= i \left\langle \alpha \left| -i [Q, \delta\Psi]_+ \right| \beta \right\rangle \\
&= 0
\end{aligned}
$$

which implies

$$\langle \alpha | Q = Q | \beta \rangle = 0. \tag{24}$$

We have thus shown that the BRST charge select physical states by means of (24).
We conclude observing that in all this procedure and derivation we never employ the canonical formalism any way. This makes us free to avoid all of the ambiguities which follow from the canonical quantization of systems with infinite degrees of freedoms [4].

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