# REVIEW ARTICLE: Quantization of constrained systems

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**Summary.** — The geometrical interpretation of first- and second-class constraints in the phase space is outlined, with the aim to demonstrate the different reduction of degrees of freedom they produce. Furthermore, the quantization of such constrained systems is analyzed, which provide a demonstration of how the Fadeev-Popov determinant arises in the path-integral formulation.

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## 1. – Geometrical interpretation of constraints

The characterization of a set of constraints being first- or second-class is based on their Poisson brackets, restricted to the constraint hypersurface [1]. However, such a procedure could not be so easily implemented. For instance, let us consider a second-class set of constraints, *i.e.*

$$\{\chi_\alpha\}, \quad [\chi_\alpha, \chi_\beta] \approx C_{\alpha\beta} \tag{1}$$

$C_{\alpha\beta}$ being an invertible matrix. The same hypersurface in the phase space can be defined by the set $\{\chi^2_\alpha\}$, which satisfy

$$[\chi^2_\alpha, \chi^2_\beta] \approx 0 \tag{2}$$

and thus it appears as a first-class set. This contradiction stands on the fact that $\{\chi^2_\alpha\}$ does not satisfy the regularity condition, which states that given a set of constraints, the matrix $\frac{\partial\chi_\alpha}{\partial\{q,p\}}$ must have maximum rank. The regularity condition itself is often difficult to verify and it does not give any insight on the meaning of constraints being first- or

 

second-class. Therefore one is looking for a different way to classify sets of constraints, based on properties of the constraints hypersurface.
Let us consider an hypersurface $\Sigma$ in the phase space, given by a parametrization $x^\mu = x^\mu(y^i)$, $(\mu = 1, \ldots, 2N)$, $y^i$ $(i = 1, \ldots, 2N - M)$ being coordinates on the hypersurface. A phase space is characterized by Poisson brackets among variables

$$[x^\mu, x^\nu] = \sigma^{\mu\nu} \tag{3}$$

where $\sigma^{\mu\nu}$ is non-degenerate, such that one can define its inverse $\sigma_{\mu\nu}$, the so-called symplectic form. Given phase space functions $F$ and $G$, a vector $G^\mu$ representing the action of one of them, for instance $G$, can be defined as $[F, G] = G^\mu \partial_\mu F$. On $\Sigma$ one can define the induced symplectic form $\sigma_{ij} = \sigma_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}$. Hence a set of constraints can be characterize in terms of properties of the induced symplectic form.

*First-class constraints*. For a set $\gamma_\alpha$ $\alpha = 1, \ldots, M$ of first-class constraints, we indicate corresponding vectors with $X^\mu_\alpha$ and vectors normal to $\Sigma$ with $n_{\mu\beta} = \nabla_\mu \gamma_\beta$. Constraints being $1^{st}$-class, we have $X^\mu_\alpha n_{\mu\beta} = 0$. Therefore $X^\mu_\alpha$ belong to the tangent space to $\Sigma$ and they can be taken as part of the basis. Moreover, they define a sub-manifold, where the induced symplectic form is degenerate, *i.e.* $\sigma_{\alpha\beta} = \sigma_{\mu\nu} X^\mu_\alpha X^\nu_\beta$ is non-invertible. One can go over and demonstrates that no other sub-manifold exists on which the induced symplectic form is degenerate.
Previous statements provide a geometrical characterization for the constraints hypersurface in the $1^{st}$-class case: *the symplectic form induced on* $\Sigma$ *is maximally degenerate.*
Therefore, in order to define a symplectic structure on constraint hypersurface, the orbits generated by constraints themselves must be factored out (reduced phase space). These orbits correspond to gauge orbits in standard gauge theories.
Finally, in order to give an Hamiltonian formulation for a system with a set of first class constraints, one not only has to restrict to the hypersurface defined by constraints, but another reduction of phase space variables has to be performed. This explains why, given a set of $M$ first-class constraints, the number of degrees of freedom is reduced to $2N - 2M$.

*Second-class constraints*. Let us now consider a set of $2^{nd}$-class constraints $\chi_\alpha$. From considerations of the previous paragraph, it is easy to recognize that vectors $X^\mu_\alpha$, giving the flux generated by constraints, do not belong to the space tangent to $\Sigma$. Furthermore, they can be taken as basis vectors for the $M$-dimensional manifold obtained by factoring out the constraint hypersurface $\Sigma$. On this manifold, the induced symplectic form is precisely $C_{\alpha\beta}$. Being Poisson brackets independent on the frame, we choose as coordinates on the full phase space the set $\{y^i, \chi_\alpha\}$. Hence, given two functions $F$ and $G$, we write their brackets on $\Sigma$

$$[F, G]|_\Sigma = \sigma^{ij} \partial_i F|_\Sigma \partial_j G|_\Sigma + C^{\alpha\beta} \partial_\alpha F|_\Sigma \partial_\beta G|_\Sigma \tag{4}$$

so that Poisson brackets on constraints hypersurface is given by Dirac brackets in the full phase space

$$\sigma^{ij} \partial_i F|_\Sigma \partial_j G|_\Sigma = [F, G]|_\Sigma - [F, \chi_\alpha] C^{\alpha\beta} [\chi_\beta, G]|_\Sigma. \tag{5}$$

Therefore, being the symplectic form induced on $\Sigma$ non-trivial, one must replace Poisson brackets with the Dirac ones.
For what concerns the number of degrees of freedom, fluxes generated by constraints lead functionals outside $\Sigma$, so that one must not factor out these orbits. In a more mathematical point of view, one can demonstrate that *the symplectic form* $\sigma_{ij}$ *is non-degenerate.* Therefore, the number of degrees of freedom for such a kind of dynamical systems is $2N - M$.

## 2. – Quantization of constrained systems

*First-class constraints.* The quantization procedure for constrained systems is a very peculiar point in a theory with some symmetry. Even though one should quantize true degrees of freedom, thus referring to the reduced phase space (*reduced phase space quantization*), there are several reasons to quantize in the full phase space [1]. At first the preservation of symmetries is very useful in a quantum model. For instance, particles can be classified according with irreducible representations of symmetry groups. Moreover in the reduced phase space quantization one can lost locality in space. As an example, invariant quantities in gauge theories are developed starting from holonomies along loops. Finally the reduced phase space quantization can provide a very complicated symplectic structure, such that one cannot be able to realize operators corresponding to physical quantities on the Hilbert space.
Therefore one usually refers to the Dirac approach. In this method the full set of phase space variables is quantized in a proper Hilbert space, where constraints are promoted to operators. The kinematical Hilbert space, which contains physical states, is defined as the kernel of such operators. This way, one solves constraints in the quantum setting. Such a procedure can give problems with the emergence of anomalies, *i.e.* constraint operators can be not first-class, because of some quantum corrections. In quantum Field Theories the emergence of anomalies is confirmed by the impossibility to find a regularization procedure preserving the symmetry. Moreover, even though one is able to define a scalar product on the initial Hilbert space, this could not be extensible to the kinematical one. One of the main approach in this sense is the group averaging technique [3].

*Second-class constraints.* Being second-class constraints not associated with true symmetries, they have to be eliminated before the quantization. It can be done directly by solving $\chi_\alpha = 0$, eliminating redundant variables and quantizing only coordinates on constraint hypersurface, or one can use the formulation in terms of Dirac brackets, by replacing them (not Poisson ones) with commutators of quantum operators. In both cases main difficulties are due to the implementation of non canonical commutation relations. A quantum framework where one implements second-class conditions exists and it is the path integral approach to gauge theories. In fact, one formally fixes the gauge and this procedure gives rise to the Fadeev-Popov determinant [2]. But the set of gauge constraints plus a gauge fixing condition is second-class. We here provide a demonstration that the emergence of the Fadeev-Popov determinant is due to the second-class character of the set of constraints.
Let us consider a phase space with a set of second-class constraints $\{C_\rho\} = \{G_a, \chi_\alpha\}$, $G_a$ being gauge constraints, thus first-class, while $\chi_\alpha$ are gauge fixing functionals. Their

algebra is as follows

$$[C_\rho, C_\sigma] = \begin{pmatrix} 0 & [G_a, \chi_\beta] \\ [\chi_\alpha, G_b] & [\chi_\alpha, \chi_\beta] \end{pmatrix}$$

so we have

$$\det[C_\rho, C_\sigma] = (\det[G_a, \chi_\beta])^2. \tag{6}$$

The path integral of the theory is developed on constraint hypersurface

$$Z = \int_\Sigma \Pi_i D y^i \sqrt{|det\sigma_{ij}|} e^{iS} \tag{7}$$

$y^i$ being coordinates on $\Sigma$, while $S$ is the action. Since $\chi_\alpha$ can be chosen as coordinates, one can rewrite the expression above as

$$Z = \int_\Sigma \Pi_{i\alpha} D y^i D\chi_\rho \delta(\chi_\rho) \sqrt{|\det \sigma_{jk}|} e^{iS}. \tag{8}$$

Let us now introduce arbitrary phase space coordinates $x^A$, for which

$$\Pi_A D x^A = \sqrt{|det\sigma_{jk}||\det C^{\mu\nu}|} \Pi_{i\sigma} D y^i D\chi_\rho, \tag{9}$$

we end up with the following form of the path integral

$$Z = \int_\Sigma \Pi_A D x^A \sqrt{|\det C_{\rho\sigma}|} e^{iS} = \int_\Sigma \Pi_A D x^A |\det[G_a, \chi_\beta]| e^{iS} \tag{10}$$

where we outline how the Fadeev-Popov determinant comes out.

## REFERENCES


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