# Review article:

# An introduction to spin foams

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### Abstract

Spin foams will be introduced from a geometrical and field-theoretical point of view. Drawing the analogy with the concept of "plaquettes" will allow one to outline the possibility of recognizing spin-network states as basis for the functionals of the connection. Spin foams, defined as branched surfaces, will accomplish the dual transformation that leads to a physically-equivalent description of lattice gauge theory. Particular attention will be paid to the description of physical observables in terms of the path-integral formulation within this formalism, and to the mathematical meaning of these operations. A spin-foam model for Yang-Mills theories will follow, and a background-independent spin-foam model for quantum gravity will be obtained by slightly modifying the duality map. The relation between spin-network states and the geometry of spacetime will be further investigated. In particular, covariant quantum gravity will be approached considering spin-network states as states of the gravitational field. Equivalence classes for spin foams will be established, and the interpretation of spin foams as quantum histories will be proposed.

# 1 Introduction

It will be illustrated [1] that any lattice gauge theory can be transformed in a physically-equivalent description, based on the spin-foam formalism. To this purpose, the idea of spin networks and spin foams will be introduced, and the techniques that lead to the analog of the path-integral definition of physical observables will be developed [2]. The background independence of the result will be discussed.
The previous scheme will be then applied to covariant quantum gravity [3]. In this case, the equivalence classes of spin networks that solve the physical constraints will be reviewed [4], and spin foams will be shown to be the tool by which the transition amplitudes between two states of the gravitational filed can be described [5].

# 2 Lattice Gauge Theory

A lattice $k$ consists of links and plaquettes; if an orientation is chosen for them, edges $e$ and faces $f$ are defined, respectively, as oriented links and plaquettes, but physical quantities are independent of the choice of the (arbitrary) orientation. In particular, the lattice $k$ is composed of the edges on the boundary $\partial k$ and the edges in the interior, $k^0$, so that

$$k = k^0 \cup \partial k, \tag{1}$$

and $E_k$ denotes the set of all the edges of $k$, $\{e\}$.
Connections on the lattice are applications that map the the edges into elements of a (compact Lie) gauge group $G$,

$$g : E_k \to G, \tag{2}$$

$$e \mapsto g_e, \tag{3}$$

$g_e$ being an element of the gauge group $G$, and the configuration space of the connections on $k$ is $A_k$.
Path integrals[1], such as

$$W[\phi] = \int \left( \prod_{e \in k} dg_e \right) e^{iS(g)} \phi(g), \tag{4}$$

are the quantities that describe physical information. In (4), $(\prod_{e\in k} dg_e)$ is the Haar measure on $G$, $S(g)$ is the action, and the function $\phi(g) \in \mathcal{L}^2_0(A_k)$ denotes the particular physics to be described. The action

$$S(g) = \sum_{f \in k^0} S_f \tag{5}$$

can be written as the sum of terms referred to each face $f \in k^0$ (face action): each term $S_f$ is required to be gauge invariant and to depend on the edges of the face itself only.

[1]Although in [1] the whole description is developed without specifying the choice of a Minkowskian or a Euclidean background, for our purposes it will be more convenient to depict the model in a Minkowskian frame.

Throughout this discussion, we will be interested in boundary-states amplitudes $\Omega[\phi]$, whose weighting functionals $\phi[\partial g] \in \mathcal{L}_0^2(A_{\partial k})$ depend on the group elements carried by boundary edges,

$$\Omega[\phi] := \int \left( \prod_{e \in k} dg_e \right) e^{iS(g)} \phi^*(\partial g), \tag{6}$$

i.e., the wave function of a physical state, that is the probability of obtaining a physical state from vacuum. Any lattice gauge model can be turned in a physically-equivalent description by means of the spin-foam formalism.

**Spin-network States** A spin network is an oriented graph, whose edges $e$ are labeled by irreducible representations of a gauge group $\rho_e$ (colours), and whose vertices $v$ are labeled by an intertwiner $I_v$ in the tensor product

$$\otimes_i V_{\rho_i \ out} \otimes_j V^*_{\rho_j \ in}, \tag{7}$$

where $V_\rho$ is the space of the irreps of $\rho$. The mathematical meaning of the intertwiners is mapping the operations on a group in the operations on another group. If one decomposes the space (7) in the sum of irreps, a sub-space can be found, which transforms according to the trivial representation, as it is composed of invariant vectors. In this picture,intertwiners form a map between $\otimes_i V_{\rho_i \quad out}$ and $\otimes_j V^*_{\rho_j \quad in}$. A spin-network state or functional $\Psi_S(g)$ can be associated to a spin network, such that

$$\psi_S(g) : g_e \to \rho_{g_e}, \tag{8}$$

and reads

$$\psi_S(g) = \left( \prod_{v \in k} I_v \right) \left( \prod_{e \in k} D_{\rho_e} \rho_e(g_e) \right) \tag{9}$$

where $D_{\rho_e} \equiv (dim V_{\rho_e})^{1/2}$. The correspondence (8) is not one-to-one, so that spin networks can be defined equivalent if they lead to the same spin-network functional. Spin-network states define the space $\mathcal{L}_0^2(A_k)$ of gauge-invariant functional of the connections $A_k$ : according to the Peter-Weyl theorem, the matrix elements of the irreps of a group form a basis for the functions of the Hilbert space of the $\mathcal{L}^2$ functions of the group. If an orthonormal basis $B_k$ for the intertwiners $I_v$ is chosen, i.e.

$$I^a{}_b = \frac{1}{D_{\rho_e}} \delta^a{}_b, \tag{10}$$

the basis for $\mathcal{L}_0^2(A_k)$ will be orthonormal too; in this case, the spin-network functional simply rewrites

$$\psi_S(g) = tr \left[ \rho \left( \prod_i g_{e_i} \right) \right]. \tag{11}$$

If only the edges on the boundary $\partial k$ are taken into account, $B_{\partial k}$ is the orthonormal basis for $\mathcal{L}_0^2(A_{\partial k})$. Loops are the spin networks on the edges that surround a face (the smallest graphs possible), and induce a basis for the face action (5), so that the exponential in (6) rewrites

$$e^{iS_f} = \sum_{S_f \in B_f} c_{S_f} \Psi_{S_f}, \tag{12}$$

where $c_{S_f}$ are suitable coefficients.
The loop functional is therefore the trace of the holonomy around a face in a given (irreducible) representation.

**Spin Foams** Spin foams are 2-dimensional branched surfaces that carry irreps and intertwiners: the definition of analogous to that of spin networks, but one dimension has to be added. Each branched surface $F$ is composed of its unbranched components $F_i$, so that $F = \bigcup_i F_i$.
Given a spin network $\psi$, a spin foam $F$ is an application such that

$$\forall \psi, F : 0 \to \psi, \tag{13}$$

and, given any two disjoint spin networks $\psi$ and $\psi'$, a spin foams $F$is an application such that maps the former into the latter, or, equivalently,

$$\forall \psi, \psi', F : \psi \to \psi'; \qquad F : 0 \to \psi^* \circ \psi', \tag{14}$$

where $\circ$ denotes disjoint union.
A spin foam is non-degenerate iff each vertex is the end-point of at least one edge, each edge of at last one face, and each face carries an irrep of the group $G$.
Equivalence classes cab be established for spin foams: spin foams are equivalent if one can be obtained from the other by affine transformation, subdivision or orientation reversal of the lattice.

Let's analyze in some detail how to express the path integral (6) in term of spin-foam amplitudes. The integration can be performed into two steps.
The path integrals can be integrated over $k^0$, as $\phi^*(\partial g)$ is not affected by the integration, i.e.,

$$\Omega[g] = \int_{\partial g' = g} \left( \prod_{e \in k^0} dg'_e \right) e^{iS(g')}, \tag{15}$$

and then inserted into (6), so that

$$\Omega[\phi] = \int \left( \prod_{e \in k^0} dg_e \right) \Omega(g) \phi^*(\partial g), \tag{16}$$

(15) can be expanded into spin-network states. In fact, the exponential of the action can be expanded as

$$e^{iS(g)} = \prod_{f \in k^0} e^{iS_f(g)} = \prod_{f \in k^0} \sum_{S_f \in B_f} C_{S_f} \psi_{S_f(g)}, \tag{17}$$

and, when substituted in (15), it brings the result

$$\Omega[g] = \int_{\partial g'=g} \prod_{e\in k^0} dg'_e e^{iS(g')} = \int_{\partial g'=g} \prod_{e\in k^0} dg'_e \prod_{f\in k^0} \sum_{S_f\in B_f} C_{S_f}\psi_{S_f}(g') = \\ = \sum_{\{f\}\to\{S_f\}} \int_{\partial g'=g} \prod_{e\in k^0} dg'_e \prod_{f\in k^0} C_{S_f}\psi_{S_f(g)}, \tag{18}$$

where, in the last step, the sum has been drawn out of the integral, and all the possible configurations $S_f$ for each $f$ have been taken into account. The introduction of spin foams is suggested by the need to evaluate each term of the sum,

$$\int_{\partial g'=g} \prod_{e\in k^0} dg'_e \prod_{f\in k^0} C_{S_f}\psi_{S_f(g')} \tag{19}$$

where spin networks are better organized into surfaces. In fact, two spin networks belong to the same unbranched surface $F_i$ if they share only one edge, and if this edge is not shared with any other spin network. The unbranched surfaces $F_i$ either are disconnected, or match other unbranched surfaces. In the latter case, the spin foam is defined as the branched surface $F = \cup_i F_i$.
In order to evaluate (19), two non-trivial cases can be distinguished, i.e.,

1. two loops match on one edge, and they carry the same label : the unbranched surface is defined as single-coloured;

2. more than two loops match on one edge, and Haar intertwiners ( a generalization of inertwiner defined formerly) have to be introduced.

As a result, all the elements contribute to the sum as follows

1. for each vertex, a factor $A_v$;

2. for each single-coloured component, a factor $\prod_i A_{F_i}$, where $A_{F_i} \propto \prod_{f\in f_i} C_{f_\rho}$ and $C_{f_\rho} \propto C_{s_f}$, the proportionality factor being a suitable power of $dimV_\rho$;

3. for each branching graph $\Gamma_F$, the projection properties of the Haar intertwiners have to be taken into account: as a result, for each vertex of the branching graph, one has to sum over all the possible ways to assign an intertwiners to the links of $\Gamma_F$.

Collecting all the terms together, one obtains

$$\Omega[g] = \sum_{F\subset k} \left( \prod_{v\in\Gamma_F} A_v \right) \left( \prod_i A_{F_i} \right) \psi_{S_f(g)}. \tag{20}$$

The product

$$\prod_{f\in F_i} C_{f_\rho} \tag{21}$$

in general depends on the discretization, and only in particular cases a geometrical interpretation is possible.

The insertion of (20) in (16) gives the final expression of the path integral $\Omega[\phi]$. If one expands $\phi(g)$ in terms of the orthonormal basis of spin networks,

$$\phi(g) = \sum_{S\in B(\partial k)} \phi_S \psi_S(g), \tag{22}$$

(16) reads

$$\Omega[\phi] = \sum_{F\subset k} \left( \prod_{v\in\Gamma_F} A_v \right) \left( \prod_i A_{F_i} \right) \phi^*_{S_F}, \tag{23}$$

i.e., the only non-vanishing contributions are brought by boundary spin networks, and each spin-foam amplitude is weighted by the coefficient of the corresponding boundary state. The comparison between the path-integral formulation (6) and (23) is eventually accomplished by noticing that the integration over connection is replaced by the sum over spin foams, and the spin-foam amplitudes wighted by the boundary functional $\phi_{S_F}$ play the role of the invariant measure and the exponential of the action, with a boundary weighting coefficient $\phi(g)$.

**Background independence** The mismatch between the idea of background independence and the geometrical interpretation of spin-foam models can be analyzed by considering two possibilities:

1. spin foams can be identified with the entire lattice, which plays the role of a discrete space-time;
2. spin foams can be interpreted as lattice-independent geometrical objects, which live on the lattice itself. The lattice, in this case, is considered as an auxiliary field, which has to be removed in the definitive model.

In the second case, the amplitudes described in the initial model must depend on the geometry of the spin foams only, i.e., in the sum

$$\Omega_k[\phi] = \sum_{F\in k} \left( \prod_{v\in\Gamma_F} A_v \right) \left( \prod_i A_{F_i} \right) \phi^*_{S_f}, \tag{24}$$

each factor $A$ depends on the branching graph only. The sum in (24) can be extended to a background-independent sum over all the equivalence classes of spin foams $F$ on a given manifold $M$, i.e., $\sum_{F\in k} \rightarrow \sum_{F\in M}$, so that

$$\Omega_k[\phi] = \sum_{F\in M} \left( \prod_{v\in\Gamma_F} A_v \right) \left( \prod_i A_{F_i} \right) \phi^*_{S_f}, \tag{25}$$

where abstract (or topological) spin foams are defined by means of abstract spin-network states, the equivalence class of spin-networks states, invariant under homeomorphisms of the boundaries. The extension (25) is possible only by the modification of the Hilbert space, as spin networks are not defined on the boundaries. $\partial k$. The new space of boundary states $\mathcal{H}$ is defined as

$$\mathcal{H}_{\partial M} = \left\{ \sum_i a_i S_i : \qquad a_i \in C, S_i \subset M, n \in N \right\}, \tag{26}$$

i.e., a finite combination of spin-network states, endowed with the structure of scalar product

$$< S, S' >= \delta_{SS'}, \tag{27}$$

for which the dual space $\mathcal{H}^*_{\partial M}$ is defined, as usual, as

$$\mathcal{H}^*_{\partial M} = \{\phi\} : \qquad \mathcal{H}_{\partial M} \to C. \tag{28}$$

The definition of such a Hilbert space is followed by the problem of overcounting, due to the homomorphisms $h$ of the manifold $M$,

$$h : M \to M, \qquad A(h^* M) = A(M), \qquad \phi_{S_{h^* F}} = \phi_{S-F}, \tag{29}$$

which can be gauged away á la Faddeev-Popov. The result is the definition of abstract or topological spin foams, and the corresponding spin-network states are the equivalence class of spin-network states invariant under homeomorphisms of the boundaries.

# 3 Spin networks and spin foams in qunatum gravity

Spin foams can be applied to covariant quantum gravity, where they play the role of the path integral, as the tool that connects different gravity states (geometries) in time. In particular, an equivalence class of 3-geometries $_i$ on a 3-d hypersurface $S_i$ is represented by a spin-network state, and the history between two different states is the spin-foam amplitude, i.e.,

$$< h_2, S_2 | h_1, S_1 >= \int_{g/g(S_1=h_1), g/g(S_2)=h_2} Dg e^{i I_{EH}(g)}, \tag{30}$$

where the measure $Dg$ is aimed at outlining the conceptual analogy with Feynman's approach rather than at defining any specific integration measure, which will be explicitly given, when needed, throughout the calculations.

Of course, the composition of spin foams must be defined, such that the transition between two states is independent of the intermediate states among which the transition is decomposed, i.e., in the sum

$$< h_3, S_3 | h_1, S_1 >= \sum_{h_2} < h_3, S_3 | h_2, S_2 >< h_2, S_2 | h_1, S_1 >, \tag{31}$$

the intermediate states 2 must carry a trivial representation, in the sense specified in the previous paragraphs. As the probability of creating a state from the vacuum, a spin foam is defined as

$$| h_1, S_1 >= \int_{g/g(S_1)=h_1} Dg e^{i I_{EH}(g)}. \tag{32}$$

**Spin-network states in canonical quantum gravity** Spin-network states can be defined following a procedure which is slightly different from the previous one, in order to realize how the geometrical properties of the state fit the constraints of the ADM formulation [4].
In particular, the Gauss, diffeomorphism and Hamiltonian constraint are encoded in the definition of the space of the physical states, annihilated by these constraints. The characterization of the physical states will be achieved by adding requests, step by step, on the properties of the general structures ( such as links and vertices) introduced in the previous paragraphs.
The SU(2) Gauss constraint annihilates those states, called cylindrical functions, defined in the Hilbert space $\mathcal{H}_{aux}$, endowed with an orthonormal basis, where linear combination and inner product are defined. In particular, cylindrical functions are $C^0$ functions on the space of the connections, which can be made invariant under SU(2) transformations by the choice of invariant intertwiners. Links that carry a representation of the SU(2) group, vertices where links intersect, and intertwiner which map the operations in the tensor product of the Hilbert spaces of the representations carried by the links define the spin-network states.
The diffeomorphism constraint annihilates those states, called s-knots, which are abstract spin-network states, defined in the Hilbert space $\mathcal{H}_{diff}$. The Hilbert space $\mathcal{H}_{diff}$ can be obtained from $\mathcal{H}_{aux}$ by considering the invariance under diffeomorphisms. This way, s-knots are invariant under both diffeomorphisms and SU(2) gauge transformations.

**Spin foams and the Hamiltonian constraint** In the canonical formulation, from a quantum-mechanical point of view, the Hamiltonian operator $H_{N,\vec{N}}(t)$, composed of the Hamiltonian and the diffoemorphism constraint, $H_{N,\vec{N}}(t)=C[N(t)]+C[\vec{N}(t)]$, can be interpreted as the generator of quantum evolution from the initial hypersurface $\Sigma_i(t=0)$ to the final hypersurface $\Sigma_f(t=1)$, parametrized by the proper time evolution $U(T)$,

$$U(T)=\int_T dNd\vec{N}U_{N,\vec{N}}=\int_T dNd\vec{N}e^{-i\int_0^1 dtH_{N,\vec{N}}(t)}. \tag{33}$$

The evolution operator $U(T)$ encodes the dynamics of the gravitational field, and its expansion in powers of $T$ can be shown to be finite order by order. The calculation of the matrix element of such an operator between two states of the gravitational field is strictly analogous to that followed in the familiar calculation of the S-matrix elements in a gauge theory, and reads explicitly

$$< s_f|U(T)|s_i >=< s_f|s_i > +(-iT)\left(\sum_{\alpha\in s_i} A_\alpha(s_i) < s_f|D_\alpha|s_i > +\sum_{\alpha\in s_f} A_\alpha(s_f) < s_f|D^+_\alpha|s_i >\right)+ \tag{34}$$

$$+\frac{(-iT)^2}{2!}\sum \alpha\in s_i \sum_{\alpha'\in s'} A_\alpha(s_i)A_{\alpha'}(s') < s_f|D_{\alpha'}|s' >< s'|D_\alpha|s_i > +... \tag{35}$$

Such a result can be obtained by splitting the calculation in several steps, i.e.,

1. the evolution from the initial hypersurface to the final one is expressed as a sum over intermediate hypersurfaces, as sketched in (31). The intermediate hypersurfaces differ by a small coordinate time, and the time evolution between two hypersurfaces can be written in terms of the diffeomorphism that describes the shift between them, so that $U_{N,\vec{N}} = D(g)U_{\vec{N},0}$;

2. the expansion of $U_{\vec{N},0}$ and the insertion of the identical projector where needed leads to a sum, where, at each order $n$, the operator $D$ acts $n$ times. Its action on the states is given by the coefficients $A$, which can be evaluated in terms of the explicit form of the Hamiltonian constraint;

3. $U(T)$ can be worked out of $U_{\vec{N},0}$ after integrating over the lapse and the shift. The first integration follows directly, as the integrand does not depend on $N$, and the second one corresponds to the implementation of the diffeomorphism constraint.

As a result, the matrix elements of the operator $U$ read as (35), where generic spin-network states have been substituted with the corresponding s-knots states. Analyzing the geometrical meaning of the intermediate states, in which the sum has been split up, allows one to recognize (35) as a sum over spin foams. In fact, the time evolution of a generic surface $s_i$ describes a "cylinder", whose time slicing are spin-network states belonging to the same s-knot, unless any interaction occurs. When operating on such a state, the Hamiltonian constraint generates a new state with one new edge and two new vertices, i.e., this structure is the elementary interaction vertex of the theory, as suggested by the comparison with ordinary gauge theories. As a generalization, the Hamiltonian constraint acts adding one dimension to the spin-network state, and such a new direction can be interpreted as time, because of the geometrical construction of (35), thus opening the way for the interpretation of these new states as spin foams. In fact, at the n-th order of sum, n new dimensions are added, and the sum can be written as the sum of topologically inequivalent term, where the weight of each vertex is determined by the coefficient of the Hamiltonian constraint. Furthermore, if the irreducible representation of the gauge group carried by the edge of each spin network is taken into account during the addition of the new vertices, the resulting geometrical objects fit the definition of spin foams given in the previous paragraph, i.e., the implementation of the Hamiltonian constraint leads naturally to the sum over spin foams.

# References


[1] F.Conrady, Geometric spin foams, Yang-Mills theory and background-independent models, gr-qc/0504059;

[2] see, for example, P. Caressa, http://www.caressa.it/matematica.html#Metodi-Matematici-della-Meccanica-Quantistica;

[3] D.Oriti, Rept.Prog.Phys. 64 (2001) 1489-1544, gr-qc/0106091;

[4] T. Thiemann, 'Introduction to Modern Canonical Quantum General Relativity', gr-qc/0110034;

[5] M.P. Reisenberger, C. Rovelli, Phys.Rev. D56 (1997) 3490-3508, gr-qc/9612035.