Gauge theories as constrained hamiltonian systems

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January 4, 2008

Abstract

When treated with an hamiltonian formalism, gauge theories behave as constrained theories, where conditions between the canonical variables hold. At first two kinds of constraints can be recognized: primary and secondary constraints. The former appearing with Lagrange multipliers, the latter arising from the consistency conditions, i.e. time indipendence, of the others. This distinction, however, is not essential, and the more usefull classification based on the Poisson algebra is adopted. This way it is possible to discard the Poisson bracktes and adopt the Dirac ones, constructed with 2nd class constraint algebra matrix $\{\phi_a, \phi_b\}$, and embed these constraints in the inner structure of the theory. Now one gets a theory with at most 1st class constraints, that generate the gauge transformations.

1 Gauge theories and primary constraints

A gauge theory is characterized by the presence of an internal group of continous symmetries, that define a inner reference system, whose choice is totally arbitrary, at every instant of time. Only quantities indipendent on that choice, i.e. gauge invariant, will be observable.
Such arbitrariness will lead to a non uniqueness in the time evolution of variables: given a set of initial conditions, after a time interval t they will be defined modulo a gauge transformation. That means that the general solution of the equation of motion will contain arbitrary function of time: in the hamiltonian formalism it corresponds to the introduction of constraints on the canonical variables.

Euler-Lagrange equation of motion can be deduced from the action S_L

$$S_L = \int_{t_0}^{t} dt L(q, \dot{q}) \tag{1}$$

and are the well known:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^n}\right) - \frac{\partial L}{\partial q^n} = 0, \tag{2}$$

that can be written as:

$$\ddot{q}^{n'} \frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n} = \frac{\partial L}{\partial q^n} - \dot{q}^{n'} \frac{\partial^2 L}{\partial q^{n'} \partial \dot{q}^n}. \tag{3}$$

From equations (3) one can see that accelerations can be expressed by position and velocities only if the matrix $\partial^2 L/\partial \dot{q}^{n'} \partial \dot{q}^n$ is invertible, i.e. its determinant is not zero.
If not the equation of motion would contain arbitrary functions of time and they will lead to a constrained system, as will be seen later.
By defining the conjugate momenta of the q's, one sees that the condition on zero determinant is actually a condition on the non invertibility of the relation $p = p(q, \dot{q})$. That means that the p's are not all indipendent, but there are relations

$$\phi_m(q, p) = 0. \tag{4}$$

These conditions are called *primary constraints* of the theory, stressing that no equation of motion was used to obtain them. In the phase space they define a (smooth) manifold called *primary constraints surface*, whose dimension is the number of indipendent relations between the (4).
Again from (4) one can see that if the inverse transformation from p to $\dot{q}$ is not biunique, a point $(q, \dot{q})$ in the 2N-dimensional phase space is projected on a lower dimensional manifold, i.e. the constraints surface, losing information about original position (see figure **??**).

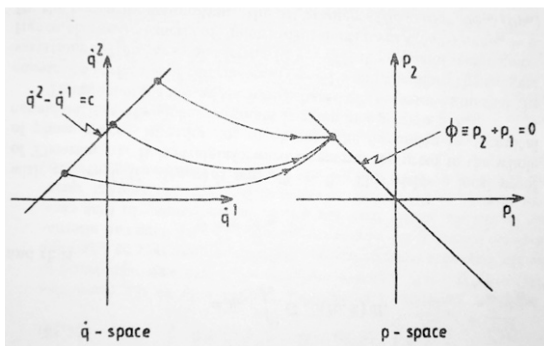

Figure 1: Map from the $(q, \dot{q})$ space to the (q, p) space.

In order to have a one to one map one has to introduce extra variables, able to preserve information about the original position in the $(q, \dot{q})$ space.

2 Hamiltonian Formalism

The canonical hamiltonian is defined with the Legendre trasformation:

$$H(q, p) = \dot{q}^n p_n - L. \tag{5}$$

In the case of a constrained theory the p's will not be independent among eachother, and they would have to verify equations (4) when they are expressed as functions of $(q, \dot{q})$.
That means that the hamiltonian is well defined just on the constraints surface, but it can be extended outside that simply adding a linear combination of the (4), with coefficients functions of the q's and the p's, that does not modify the dynamics:

$$H \rightarrow H + c^m(q, p)\phi_m. \tag{6}$$

By variation of the equation (5 one gets the equality:

$$\frac{\partial H}{\partial q^n}\delta q^n + \frac{\partial H}{\partial p^n}\delta p^n = \delta H = \dot{q}^m \delta p_m - \delta q^m \frac{\partial L}{\partial q^m}.$$

Rearranging:

$$\left(\frac{\partial H}{\partial q^m} + \frac{\partial L}{\partial q^m}\right)\delta q^m + \left(\frac{\partial H}{\partial p^m} - \dot{q}^m\right)\delta p_m = 0. \tag{7}$$

Using a theorem stating that if $\lambda_n \delta q^n + \mu \delta_n p^n = 0$ for arbitrary variations of the canonical variables then:

$$\lambda_n = u^m \frac{\partial \phi_m}{\partial q^n}, \tag{8}$$

$$\mu_n = u^m \frac{\partial \phi_m}{\partial p^n} \tag{9}$$

for some u^m, one can obtain a biunique expressione for the $\dot{q}$'s and so recover a one-to-one mapping between the $(q, \dot{q})$ and (q, p) spaces. There is a price to pay: one has to introduce some extra variables u^m.

This way is possible to rewrite the equations of motion (2) in the hamiltonian form, using variables p, q and u:

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \tag{10}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n}, \tag{11}$$

$$\phi_m(q, p) = 0. \tag{12}$$

Such procedure will modify the variational principle (1) from which these equations are obtained, adding the extra variables u:

$$\delta \int_{t_0}^{t} dt(\dot{q}^n p_n - H - u^m \phi_m) = 0 \tag{13}$$

that appear as Lagrange multipliers for the primary constraints. It is obvious that such theory is invariant under transformations like $H \to H + c^m(q, p)\phi_m$, which give just a new definition of the u^m's.
It is now straight-forward to write down the equation of motion in the hamiltonian formalism, by using the Poisson brackets:

$$\dot{F} = \{F, H\} + u^m\{F, \phi_m\}, \tag{14}$$

defined as:

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}. \tag{15}$$

3 Secondary constraints and total hamiltonian

The condition of consistency for the theory is that the primary constraints must be preserved by time evolution. That means, using the Hamiltonian function as time translation generator:

$$\{\phi_m, H\} + u^{m'}\{\phi_m, \phi_{m'}\} = 0. \tag{16}$$

Considering the case in which one of the resulting relations is independent from the u^m's, one may obtain again one of the primary constraints, even

expressed in a different form, or a new relation between the q's and the p's. That will be called a *secondary constraint*.
However, secondary constraints must satisfy the consistency conditions (16) too. That may result in others secondary constraints. By iteration one will obtain K secondary constraints, completing the set:

$$\phi_j = 0, \qquad j = 1, ..., M + K = J \tag{17}$$

One can now analyze restrictions imposed on the u^m's by equations (16)[1]:

$$\{\phi_j, H\} + u^m\{\phi_j, \phi_m\} \approx 0, \qquad m = 1, ...M \quad j = 1, ..., J \tag{18}$$

and consider them as J non-homogeneous equations in the $M \leq J$ variables u^m, with coefficients that are functions of p, q, which must have solutions to make the theory stand.
The generic solution will have the form:

$$u^m = U^m + V^m \tag{19}$$

where U^m is the particular solution of the non-homogeneous equation and V^m is the general solution of the associated homogeneous equation:

$$V^m\{\phi_j, \phi_m\} \approx 0. \tag{20}$$

The most generic form for V^m is a linear combination of independente solutions V_a^m, with $a = 1, ..., A$, where A is constant for all p and q if one assumes a constant rank for the matrix $\{\phi_j, \phi_m\}$.
A solution of (18) is obtained:

$$u^m \approx U^m + v^a V_a^m \tag{21}$$

with the coefficients v_a that are *totally arbitrary*. One has so achieved a separation between these u^m's, which are fixed by consistency conditions (16) and are totally undetermined.

It is now possible, using that result, to rewrite the equations of motion as:

$$\dot{F} \approx \{F, H' + v^a\phi_a\} = \{F, H_T\}, \tag{22}$$

defining the *total hamiltonian* function of the system $H_T = H' + v^a\phi_a$, with $H' = H + U^m\phi_m$, equivalent to the (2)'s.

4 First and second class contraints

A function $F(q, p)$ is said to be *first class* if its Poisson brackets with every constraint are weakly vanishing:

$$\{F, \phi_j\} \approx 0, \qquad j = 1, ..., J \tag{23}$$

Otherwise it is said *second class*. Fundamental property of firts class functions is that even Poisson brackets among themselves are first class.
In particular H' and ϕ_a are first class. Furthermore the ϕ_a's are a base in the first class constraints space, because $v_a V^a$ is the most general solution of the homogeneous equation (20). So the total hamiltonian H_T is the sum of the first class hamiltonian H' and of a linear combination of first class constraits, with arbitrary coefficients.

[1]The symbol $\approx$ is used for relations holding on the constraints surface (weak equality), the symbol $=$ is used for relations holding in the whole phase space (strong equality).

It is therefore favourable, in the hamiltonian formulation, to abandon the distinction between primary and secondary constraints in favour of the one between first and second class, whose definition is based on the fundamental structure of the hamiltonian formalism, the Poisson brackets.
One can assume, in general, that *every first class constraint is the generator of a gauge transformation.*

From now on it will be adopted a notation that distinguishes between these two kinds of constaints: γ for first class, χ for second class ones. The whole set of constraints will be named ϕ_j.
Remembering that all the γ's, primary and secondary, are gauge generators, one has to modify the hamiltonian function, which in (22) includes just the first class primary constraints. Then one introduces the *extended hamiltonian*, in the form:

$$H_E = H' + u^a \gamma_a. \tag{24}$$

It is obvious that for gauge invariant quantities, i.e. observables, dynamics can be calculated without distinction with H', H_T or H_E, because the Poisson brackets of these quantities with the generators will vanish.

5 Second class constraints and Dirac brackets

When second class constraints are present, the JxJ square matrix $C_{ij} = \{\phi_i, \phi_j\}$ will not vanish on the constraints surface.
By redefining the constraints with a coordinate transformation $a^j{}_{j'}$ one can obtain a separation between first and second class constraints, making C_{ij} a block matrix on the constraints surface:

$$\mathbf{C} = \begin{pmatrix} \{\gamma_a, \gamma_b\} & \{\gamma_a, \chi_\alpha\} \\ \{\gamma_b, \chi_\beta\} & \{\chi_\beta, \chi_\alpha\} \end{pmatrix} \approx \begin{pmatrix} 0 & 0 \\ 0 & C_{\beta\alpha} \end{pmatrix} \tag{25}$$

The second class constraints matrix $C_{\beta\alpha}$ will be antisymmetric, as its definition from the Poisson brackets asks, and its dimension will have to be even to avoid a vanishing determinant. The second class constraints will cast allowed states in unallowed states, because the condition $\phi_j \approx 0$ is not fulfilled.
Being the problem in the action of the Poisson brackets, one can define the Dirac brackets.
If the matrix $C_{\beta\alpha}$ has a non vanishing determinant it will be invertible, and one would be able to define the new brackets to be implemented in the theory, as:

$$\{F, G\}^* = \{F, G\} - \{F, \chi_\alpha\} C^{\alpha\beta} \{\chi_{beta}, G\}, \tag{26}$$

which have all the properties of the Poisson bracktes. In particular $\{\chi_\alpha . F\}^* = 0$ for all F. Secondary constraints can now be imposed before or after the calculation of the Dirac brackets, without distinction. Obviously the matrix $C_{\beta\alpha}$ is defined modulo a coordinate transformation in the 2nd class constraints space, that is a simple redefinition of them.
Furthermore, being the extended hamiltonian (24) first class, it will generate the equation of motion with the Dirac brackets too:

$$\dot{F} \approx \{F, H_E\} \approx \{F, H_E\}^*. \tag{27}$$

In the same way one can redefine gauge transformations as:

$$\{F, \gamma_a\} \approx \{F, \gamma_a\}^* \tag{28}$$

without losing generality.

So the Poisson brackets, at first used to distinguish between first and second class constraints, are dropped in favour of the Dirac brackets, that let reformulate the theory coherently, implementing the action of second class contraints in the structure of the theory itself.
One can so obtain a theory with just first class constraints, which are the generators of the gauge group(s) of the theory.