How to use Unimodular Quantum Cosmology for the Prediction of a late-time Classical Universe ?

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Abstract

Unimodular quantum cosmology admits wavepacket solutions that evolve according to a kind of Schrödinger equation. Though this theory is equivalent to general relativity on the classical level, its canonical structure is different and the problem of time does not occur. We present an Ehrenfest theorem for the long term evolution of the expectation value of the scale factor for a spatially flat Friedmann universe with a scalar field. We find that the classical and the quantum behaviour in the asymptotic future coincide for the special case of a massless scalar field. We examine the general behaviour of the uncertainty of the scale factor in order to single out models that can lead to a classical universe.

1   Introduction

The canonical quantization of general relativity leads to the so-called problem of time (see [1] and references therein). In most non-perturbative approaches of quantum gravity time has disappeared from the theory and is seen as an artifact of the classical limit. Here we investigate quantum cosmology in the framework of unimodular gravity. This theory is practically equivalent to general relativity at the classical level, but since it has a different canonical structure time does not disappear from the quantum theory ([2]) and it is possible to study the evolution of wave packet solutions and the behaviour of expectation values compared to the classical evolution of their counter parts.

In ([3]) we constructed a class of unitarily evolving solutions with a negative expectation value of the Hamiltonian for the special case of a spatially flat universe with a massless scalar field. Investigating a special example, we found that the classical and quantum dynamics of the scale factor coincide for the asymptotic future (though with significant spread). Here we compare the expectation value of the scale factor to the evolution of its classical counterpart for solutions of the spatially flat Friedmann universe with an arbitrary scalar

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field which yields Ehrenfest theorem for the late time behaviour. We examine the evolution of the uncertainty of the scale factor in order to single out models that can lead to a classical universe in the asymptotic future.

2   About Unimodular Gravity

We start with the Einstein Hilbert action (1)

$$S_{EH} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} \, (R - 2\Lambda) - \frac{1}{\kappa} \int_{\partial\mathcal{M}} d^3x \sqrt{h} \, K \,, \tag{1}$$

where

$$\kappa = \frac{8\pi G}{c^4}$$

contains the velocity of light c and the gravitational constant G. We also take into account the matter action S_m that describes the fields. If we vary the action $S = S_m + S_{EH}$ with respect to the metric $g_{\mu\nu}$ under the restriction $-g = 1$, we obtain Einsteins equations with an arbitrary additional constant Λ , that can be identified with the cosmological constant of general relativity ([2]).

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \, T_{\mu\nu} - \Lambda \, g_{\mu\nu} \tag{2a}$$

$$\sqrt{-g} - 1 = 0 \,. \tag{2b}$$

This theory is called unimodular gravity. Any solution of unimodular gravity (2) is also a solution of general relativity for a specific cosmological constant and vice versa. The only difference between the two theories is, that Λ is a natural constant in general relativity while it is a conserved quantity in unimodular gravity. But since in both theories the cosmological constant can not vary over the whole universe, we would have to investigate different universes to determine if solutions with different Λ exist (unimodular theory) or if Λ is a "true" natural constant. So the two theories are practically indistinguishable. Nevertheless the canonical structure of the theories differs ([2])and therefor the quantization of unimodular theory yields different results compared to the quantization of general relativity ([4]).

3   The Unimodular Hamiltonian of a spatially flat Friedmann Universe

The metric of a homogeneous and isotropic spacetime (Friedmann universe)

$$ds^2 = -N^2(t)c^2 dt^2 + a^2(t) d\Omega_3^2 \tag{3}$$

is characterized by the lapse function N(t) and the scale factor $a(t)$. If the spatial curvature is zero, $d\Omega_3^2$ is the line element of three-dimensional flat space.

Inserting the metric into the Einstein-Hilbert action (1) with $\Lambda = 0$ yields ([1])

$$S_{EH} = \frac{3}{\kappa} \int dt \, N \left(-\frac{\dot{a}^2 a}{c^2 N^2} \right) v_0 \,,$$

where v_0 is is the volume of the spacelike slices according to (3).

The action of a scalar field in a Friedmann universe (3) reads

$$S_m = \int dt \, N a^3 \left(\frac{\dot{\phi}^2}{2N^2 c^2} - V(\phi) \right) v_0 \,. \tag{4}$$

Using the unimodular condition for the lapse function $N = a^{-3}$, we find for the Hamiltonian ([3])
of the unimodular theory

$$H_{uni} = \frac{c^2}{2} \frac{p_\phi^2}{a^6} - \frac{c^2}{4\epsilon} \frac{p_a^2}{a^4} + V(\phi) \tag{5}$$

The Hamiltonian is a conserved quantity and not a constraint as in general relativity

The canonical quantization of this Hamiltonian yields

$$\hat{p}_a = -i\hbar \frac{\partial}{\partial a} \,, \quad \hat{p}_\phi = -i\hbar \frac{\partial}{\partial \phi} \,, \tag{6}$$

$$\widehat{H} = \frac{\hbar^2 c^2}{4\epsilon} \frac{1}{a^5} \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{\hbar^2 c^2}{2} \frac{1}{a^6} \frac{\partial^2}{\partial \phi^2} + V(\phi) \,.$$

Here we have chosen the factor ordering that gives the part of the Hamiltonian that is quadratic in the momenta the form of a Laplace Beltrami operator ([1])

The evolution of the wavefunction $\psi(a, \phi, t)$ is determined by

$$\widehat{H}\psi = i\hbar \frac{\partial}{\partial t} \psi \,. \tag{7}$$

The Hamiltonian is symmetric with respect to the inner product defined by the measure $a^5 da \, d\phi$, where $a \in (0, \infty)$ and $\phi \in (-\infty, \infty)$.

Applying the coordinate transformations

$$A = a^3/3 \qquad B = \frac{3}{\sqrt{2\epsilon}} \phi \,, \tag{8}$$

and

$$u = Ae^{-B} \qquad v = Ae^B \,. \tag{9}$$

We obtain the Hamilton operator

$$\widehat{H} = \frac{\hbar^2 c^2}{\epsilon} \frac{\partial^2}{\partial u \partial v} + V\left(\frac{u}{v} \right) \tag{10}$$

The volume element is given by $dudv$ and $u \in (0, \infty)$, $v \in (0, \infty)$.
The classical unimodular Hamiltonian then reads

$$H = -\frac{c^2}{\epsilon} p_u \, p_v + V\left(\frac{u}{v}\right) \, .$$

4 General properties of the time evolution

In [3] we have derived a class of unitarily evolving wavepacket solutions of (7)
for the special case of a massless scalar field $(V = 0)$. We found that these
solutions fulfill in the late phase of time evolution

$$\lim_{t \to \infty} \psi(0, v, t) = \lim_{t \to \infty} \psi(u, 0, t) = 0 \, . \tag{11}$$

Now we assume that a wavepacket solution for a general scalar field can be
found and that it also fulfills (11). We investigate the physical behaviour of
these solutions compared to the evolution of the classical quantities.

As in ordinary quantum mechanics, the evolution of the expectation values of
an observable $\widehat{O}$ with respect to a solution $\psi(u, v, t)$ of the Schrödinger equation
(7) is given by

$$\frac{d}{dt}\langle\psi|\widehat{O}|\psi\rangle = -\frac{i}{\hbar}\left(\langle\psi|\widehat{O}\widehat{H}|\psi\rangle - \langle\widehat{H}\psi|\widehat{O}\psi\rangle\right) \, . \tag{12}$$

This implies the equation

$$\frac{d}{dt}\langle\psi|\widehat{O}|\psi\rangle = -\frac{i}{\hbar}\left\langle\psi\left[\widehat{O}, \widehat{H}\right]\psi\right\rangle \, , \tag{13}$$

only if $\widehat{O}\psi$ obeys

$$\langle\widehat{H}\psi|\widehat{O}\psi\rangle \overset{!}{=} \langle\psi|\widehat{H}\widehat{O}|\psi\rangle \, . \tag{14}$$

In the case of our model (10), this condition reads

$$\int_0^\infty \psi^*(u, v, t)\frac{\partial}{\partial v}\widehat{O}\psi(u, v, t)dv\Big|_{u=0} - \int_0^\infty \left(\frac{\partial}{\partial u}\psi^*(u, v, t)\right)\widehat{O}\psi(u, v, t)du\Big|_{v=0} = 0 \, . \tag{15}$$

We find that this condition is fulfilled in the limit $t \to \infty$ for wavepackets
with the property (11). This means that the time evolution that we derive by
the application of (13), gives the correct result for the limit $t \to \infty$, and is
approximately valid in the late phase of time evolution.

We find for the variable uv, related to the scalefactor by

$$A^2 = uv = \frac{a^6}{9} \, ,$$

$$\lim_{t \to \infty}\frac{d^2}{dt^2}\langle A^2\rangle = \frac{c^2}{\epsilon}\left\langle -2\,\widehat{H} + V + u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v}\right\rangle \, , \tag{16a}$$

whereas the classical time evolution reads

$$\lim_{t \to \infty} \frac{d^2}{dt^2} A^2 = \frac{c^2}{\epsilon} \left(-2\,H + V + u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v} \right) . \qquad (16b)$$

So we see that in the special case of a massless scalar field the late time behaviour of the classical scalefactor and its expectation value according to unimodular quantum cosmology are the same. In general (16) represents the Ehrenfest theorem for unimodular quantum cosmology.

The result for the uncertainty $\Delta(A^2)$ in the special case of a massless scalar field reads

$$\lim_{t \to \infty} \frac{d^4}{dt^4} (\Delta(A^2))^2 = \frac{24c^2}{\epsilon^2} (\Delta H)^2 \,,$$

which implies that the uncertainty $\Delta(A^2)$ is monotonically growing with t^2 in the late phase of time evolution.

We have already first results of the evolution of the uncertainties for general scalar fields and are formulating conditions for models that yield a more and more classical universe in the asymptotic future. Work in progress!

References

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