Hamiltonian Analysis In New General Relativity

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It is known that one can formulate an action in teleparallel gravity which is equivalent to general relativity, up to a boundary term. In this geometry we have vanishing curvature, and non-vanishing torsion. The action is constructed by three different contractions of torsion with specific coefficients. By allowing these coefficients to be arbitrary we get the theory which is called "new general relativity". The Lagrangian for new general relativity is is written down in ADM-variables. In order to write down the Hamiltonian we need to invert the velocities to canonical variables. However, the inversion depends on the specific combination of constraints satisfied by the theory (which depends on the coefficients in the Lagrangian). It is found that one can combine these constraints in 9 different ways to obtain non-trivial theories, each with a different inversion formula. The teleparallel equivalent to general relativity gives 2 degrees of freedom. However, this number does not hold for arbitrary coefficients.

Keywords: Teleparallel gravity; New general relativity; ADM-variables.

1. Conventions

Greek indices denote global coordinate indices running from 0 to 3, small latin indices are spatial coordinate indices running from 1 to 3, whereas capital latin indices denote Lorentz indices running from 0 to 3. Whenever we put a bullet on top of an object we emphasize that we are in a teleparallel geometry. We are always dealing with Lorentzian metrics.

2. Introduction

Gravity is normally described as a theory on a pseudo-Riemannian manifold. This means that spacetime is curved, with Lorentzian metric, torsion-free connection and that the covariant derivative of the metric is zero. However, there are equivalent theories to general relativity. I will focus on teleparallel gravity where we have vanishing curvature, but non-vanishing torsion.

In particular I will perform the Hamiltonian analysis of "new general relativity". Previous work on the Hamiltonian analysis on teleparallel gravity theories have been performed in[1–13]. However, the full Hamiltonian analysis of new general relatvity has not been performed. New general relativity is described by the following action:

$$S_{\mathrm{NGR}} = m_{Pl}^2 \int |\theta| \left(a_1 T^{\mu}{}_{\nu\rho} T_{\mu}{}^{\nu\rho} + a_2 T^{\mu}{}_{\nu\rho} T^{\rho\nu}{}_{\mu} + a_3 T^{\mu}{}_{\rho\mu} T^{\nu\rho}{}_{\nu} \right) \mathrm{d}^4 x, \tag{1}$$

where m_{Pl} is the Planck mass,

$$T^{\mu}{}_{\nu\rho} = \overset{\bullet}{\Gamma}{}^{\mu}{}_{\nu\rho} - \overset{\bullet}{\Gamma}{}^{\mu}{}_{\rho\nu} \tag{2}$$

is the torsion component with

$$\overset{\bullet}{\Gamma}{}^{\mu}{}_{\nu\rho} = e_A{}^{\mu} \partial_\rho \theta^A{}_\nu + e_A{}^{\mu} \left(\Lambda^{-1}\right)^A{}_D \partial_\rho \Lambda^D{}_B \theta^B{}_\nu, \tag{3}$$

with θ being the tetrad, e its inverse and Λ is a Lorentz matrix. Global spacetime indices are raised and lowered with $g_{\mu\nu} = \theta^A{}_\mu \theta^B{}_\nu \eta_{AB}$, while Lorentz indices are raised and lowered by the Minkowski metric η_{AB}. The following parameters yield a theory equivalent to general relativity whose action only differ by a boundary term:

$$a_1 = \tfrac{1}{4}, \qquad a_2 = \tfrac{1}{2}, \qquad \text{and} \qquad a_3 = -1. \tag{4}$$

3. Method

In order to go from the Lagrangian to the Hamiltonian analysis we need to identify the velocities (time derivatives on the fundamental fields), derive the conjugate momenta and express everything in canonical variables. We may decompose the torsion scalar in the ADM variables[14] lapse α, shift β^i and the spatial components of the tetrad θ^A_i:

$$\begin{aligned}
\mathbb{T} = & \frac{1}{2\alpha^2} T^A_{i0} T^B_{j0} M^i{}_A{}^j{}_B \\
& + \frac{1}{\alpha^2} T^A_{i0} T^B_{kl} \left[M^i{}_A{}^l{}_B \beta^k + 2\alpha a_2 h^{il} \xi_B \theta_A{}^k + 2\alpha a_3 h^{il} \xi_A \theta_B{}^k \right] \\
& + \frac{1}{\alpha^2} T^A{}_{ij} T^B{}_{kl} \left[\frac{1}{2} M{}^i_A{}^k_B \beta^j \beta^l + 2\alpha a_2 h^{jl} \xi_A \theta_B{}^i \beta^k + 2\alpha a_3 h^{jl} \xi_A \theta_B{}^k \beta^i \right] \\
& + {}^3\mathbb{T},
\end{aligned} \tag{5}$$

where $h_{ij} = \theta^A_i \theta^B_j \eta_{AB}$ is the induced metric, which is used to raise and lower spatial indices, $\xi^A = -\frac{1}{6} \epsilon^A{}_{BCD} \theta^B_i \theta^C_j \theta^D_k \epsilon^{ijk}$,

$$M^i{}_A{}^j{}_B = -2a_1 h^{ij} \eta_{AB} + (a_2 + a_3) \xi_A \xi_B h^{ij} - a_2 \theta_A{}^j \theta_B{}^i - a_3 \theta_A{}^i \theta_B{}^j, \tag{6}$$

and

$$\begin{aligned}
{}^3\mathbb{T} \equiv\ & a_1 \eta_{AB} T^A_{ij} T^B_{kl} h^{ik} h^{jl} + a_2 \eta_{AC} \theta^C{}_m h^{im} \eta_{BD} \theta^D{}_p h^{jp} T^A_{kj} T^B_{li} h^{kl} \\
& + a_3 \eta_{AC} \theta^C{}_m h^{im} \eta_{BD} \theta^D{}_p h^{jp} h^{kl} T^A_{ki} T^B_{lj}.
\end{aligned} \tag{7}$$

Without any loss of generality[15] we can restrict ourselves to the Weitzenböck gauge for which the torsion components are expressed as:

$$T^A_{\mu\nu} = \partial_\nu \theta^A{}_\mu - \partial_\mu \theta^A{}_\nu, \tag{8}$$

and hence the conjugate momenta become:

$$\alpha \frac{\pi^i_A}{\sqrt{h}} = T^B_{j0} M^i{}_A{}^j{}_B + T^B_{kl} \left[M^i{}_A{}^l{}_B \beta^k + 2\alpha a_2 h^{il} \xi_B \theta_A{}^k + 2\alpha a_3 h^{il} \xi_A \theta_B{}^k \right]. \tag{9}$$

The velocities can now be inverted and expressed in canonical variables using:

$$S^i_A = \dot{\theta}^B{}_j M^i{}_A{}^j{}_B, \tag{10}$$

with

$$S^i_A = D_j\left(\alpha\xi^B + \beta^m\theta^B_{\ m}\right)M^{i\ j}_{\ A\ B} - T^B_{\ kl}\left[M^{i\ l}_{\ A\ B}\beta^k + 2\alpha a_2 h^{il}\xi_B\theta_A^{\ k} + 2\alpha a_3 h^{il}\xi_A\theta_B^{\ k}\right] + \alpha\frac{\pi^i_A}{\sqrt{h}}, \tag{11}$$

where D_i is the Levi-Civita covariant derivative with respect to the induced metric. However, M in equation (10) is singular for certain combinations of parameters of the theory and can hence only be inverted by the Moore-Penrose pseudo-inverse matrix[5]. This is apparent if one decomposes the equation into irreducible representations of the rotation group, which generates the following constraints:

$$2a_1 + a_2 + a_3 = A_1 = 0 \implies {}^VC^i = S^i_A\xi^A = 0, \tag{12}$$

$$2a_1 - a_2 = A_2 = 0 \implies {}^AC_{ij} = S^k_A\theta^A_{\ [j}h_{i]k} = 0, \tag{13}$$

$$2a_1 + a_2 = A_3 = 0 \implies {}^SC_{ij} = S^k_A\theta^A_{\ (j}h_{i)k} - \frac{1}{3}S^k_A\theta^A_{\ k}h_{ij} = 0, \tag{14}$$

$$2a_1 + a_2 + 3a_3 = A_4 = 0 \implies {}^TC = S^i_A\theta^A_{\ i} = 0. \tag{15}$$

These are primary constraints since these constrain both the tetrad field and their conjugate momenta, which also can be decomposed into irreducible parts.

4. Results

There are 9 non-trivial classes of theories, which are made by imposing a combination of (12)-(15):

Theory	Implied constraints
$A_i \neq 0\ \forall i\ \in\ \{1,2,3,4\}$	No contraints
$A_1 = 0$	${}^VC_i = 0$
$A_2 = 0$	${}^AC_{ji} = 0$
$A_3 = 0$	${}^SC_{ji} = 0$
$A_4 = 0$	${}^TC = 0$
$A_1 = A_2 = 0$	${}^VC_i = {}^AC_{ji} = 0$
$A_2 = A_3 = 0$	${}^SC_{ji} = {}^AC_{ji} = 0$
$A_2 = A_4 = 0$	${}^AC_{ji} = {}^TC = 0$
$A_1 = A_3 = A_4 = 0$	${}^VC_i = {}^SC_{ji} = {}^TC = 0$

The Hamiltonian is found to always appear with four Lagrange multipliers (linearity in lapse and shifts) with:

$$H = \alpha\mathcal{H}\left(\theta^A_{\ i}, \left(M^{-1}\right)_{i\ k}^{\ A\ C}\right) + \beta^k\mathcal{H}_k\left(\theta^A_{\ i}, \left(M^{-1}\right)_{i\ k}^{\ A\ C}\right) + D_i\left[\left(\alpha\xi^A + \beta^j\theta^A_{\ j}\right)\pi^i_A\right], \tag{16}$$

in the unconstrained case. This gives rise to 4 secondary constraints. In addition 4 degrees of freedom in phase-space can be removed by diffeomorphism invariance, 8 by the primary constraints for $\theta_A^{\ 0}$ and then some more primary constraints corresponding to the constraints of the specific theory. These degrees of freedom are subtracted from the initial 32 degrees of freedom corresponding to the tetrad components and their conjugate momenta. In teleparallel equivalence to general relativity there are as expected 4 degrees of freedom in phase space and the Poisson brackets are all shown to be of first class in[1–5].

5. Discussion

It is apparent that we can get different number of degrees of freedom depending on which class of theory we are in. The degrees of freedom are consistent with what we would expect from general relativity. To properly calculate the number of degrees of freedom, one needs to calculate the Poisson brackets between the constraints and the Hamiltonian (seperately for each of the 9 cases).

However, if one assumes that all Poisson brackets are of first class, then the counting for each case can be performed and this also gives us a lower limit on the number of degrees of freedom. The number of degrees of freedom can be compared with polarization modes in gravitational waves. One may extend this analysis to $f(T_{\rm NGR})$ or include parity violating terms. Another direction to proceed in is to make a teleparallel gravity formulation in other curvature based modified gravity theories. Then the counting of degrees of freedom has to be well understood.

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