# Quantization of the Szekeres Spacetime

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We present the effect of the quantum corrections on the Szekeres spacetime, a system important for the study of the anisotropies of the pre-inflationary era of the universe. The study is performed in the context of canonical quantisation in the presence of symmetries. We construct an effective classical Lagrangian and impose the quantum version of its classical integrals of motion on the wave function. The interpretational scheme of the quantum solution is that of Bohmian mechanics, in which one can avoid the unitarity problem of quantum cosmology. We discuss our results in this context.

*Keywords*: Szekeres system; Silent universe; Quantization; Semiclassical approach

## 1. Introduction

The interest on the silent universe lies on the fact that they can be seen as inhomogeneous models rendering them proper for the description of FLRW spacetimes perturbations[1–5]. Indeed, it is known that there exists a family of exact solutions for the field equations of the silent universe described as Szekeres geometries with line element of the form[6]

$$ds^2 = -dt^2 + e^{2\alpha} dr^2 + e^{2\beta} \left( dy^2 + dz^2 \right) \tag{1}$$

where $\alpha \equiv \alpha(t, r, y, z)$ and $\beta \equiv \beta(t, r, y, z)$. These correspond to the Friedmann–Lemaître–Robertson–Walker-like geometries and the Kantowski-Sachs solutions[7] in which the two components of the electric part of the Weyl tensor and the two components of the shear for the observer $u^{\mu}$ are equal. The field equations for the silent universe reduce to a system of algebraic-differential equations

$$\frac{\theta^2}{3} - 3\sigma^2 + \frac{{}^{(3)}R}{2} = \rho, \tag{2a}$$

$$\dot{\rho} + \theta\rho = 0, \tag{2b}$$

$$\dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2}\rho = 0, \tag{2c}$$

$$\dot{\sigma} - \sigma^2 + \frac{2}{3}\theta\sigma + E = 0, \tag{2d}$$

$$\dot{E} + 3E\sigma + \theta E + \frac{1}{2}\rho\sigma = 0, \tag{2e}$$

where $\dot{}$ denotes the directional derivative along $u^\mu$, the energy density is $\rho = T^{\mu\nu}u_\mu u_\nu$, with $T^{\mu\nu}$ being the energy-momentum tensor of the matter, the parameter $\theta = (\nabla_\nu u_\mu)\, h^{\mu\nu}$ is the expansion rate of the observer, while $\sigma$ and $E$ are the shear and electric component of the Weyl tensor, $E^\mu_\nu = Ee^\mu_\nu$, $\sigma^\mu_\nu = \sigma e^\mu_\nu$, in which the set of $\{u^\mu, e^\mu_\nu\}$ defines an orthogonal tetrad.

In the following we study the quantization of this system in terms of canonical quantization in the presence of symmetries [8]. The starting point is the effective Lagrangian obtained in [9]. The physical properties at the quantum level are studied by adopting the Bohmian interpretation [10,11] since it is well suited for quantum cosmology [12,13].

## 2. Classical and Quantum Dynamics

In [9] the Szekeres system (2) was written in an equivalent form of a two second-order differential equations system

$$\ddot{x} + 2\frac{\dot{y}}{y}\dot{x} - \frac{3}{y^3}x = 0, \tag{3a}$$

$$\ddot{y} + \frac{1}{y^2} = 0. \tag{3b}$$

where the variables $x, y$ are related to the energy density and the electric term as $\rho = \frac{6}{(1-x)y^3}$, $E = \frac{x}{y^3(x-1)}$, while the expansion rate and the shear are defined by the equations (2b), (2e) as $\theta = -\frac{\dot{\rho}}{\rho}, \sigma = \frac{2\left(\dot{\rho}E - \rho\dot{E}\right)}{\rho(\rho+6E)}$. It was shown there that the dynamical system (3) can be derived by a variational principle with Lagrange function [9]

$$L\left(x, \dot{x}, y, \dot{y}\right) = y\dot{x}\dot{y} + x\dot{y}^2 - xy^{-1} \tag{4}$$

The system (3) admits two integrals of motion, quadratic in the velocities; the first is the Hamiltonian function since the system is autonomous, while the second one is the quadratic function $I_0$ which can be constructed by the application of Noether's theorem for contact symmetries [9]. Choosing a new set of variables $\{u, v\}$ defined by $x = vu^{-1}, y = u$ and turning to the phase space, the Hamiltonian and the conserved quantity $I_0$ can be written in terms of the momenta as

$$p_u p_v + \frac{v}{u^2} = h, \tag{5a}$$

$$p_v^2 - 2u^{-1} = I_0 \tag{5b}$$

When turned to quantum operators and imposed on the wave function lead to two eigenvalue equations

$$\left(-\partial_{uv} + \frac{v}{u^2}\right)\Psi = h\Psi, \tag{6a}$$

$$\left(\partial_{vv} + \frac{2}{u}\right)\Psi = -I_0\Psi, \tag{6b}$$

the first one being the time-independent Schroödinger equation. Their solution is

$$\Psi\left(I_0, u, v\right) = \frac{\sqrt{u}}{\sqrt{2 + I_0 u}}\left(\Psi_1 \cos f\left(u, v\right) + \Psi_2 \sin f\left(u, v\right)\right) \tag{7}$$

where

$$f\left(u, v\right) = \frac{(hu + I_0 v)\sqrt{2I_0 + I_0^2 u} - 2h\sqrt{u}\,\text{arcsinh}\sqrt{\frac{I_0 u}{2}}}{I_0^{3/2}\sqrt{u}}, \quad \text{for } I_0 \neq 0, \tag{8}$$

$$f\left(u, v\right) = \frac{\sqrt{2}\left(hu^2 + 3v\right)}{3\sqrt{u}}, \quad \text{for } I_0 = 0. \tag{9}$$

where the coefficients $\Psi_1$ and $\Psi_2$ are constants of integration. Due to the linearity of (6a), the general solution is $\Psi_{Sol}\left(u, v\right) = \sum_{I_0} \Psi\left(I_0, u, v\right)$.

## 3. Semiclassical analysis and probability

In the context of Bohmian mechanics, the departure from the classical theory is determined by an additional term in the classical Hamilton-Jacobi equation, known as quantum potential $Q_V = -\frac{\Box\Omega}{2\Omega}$, where $\Omega$ denotes the amplitude of the wave function in polar form, $\Psi(u, v) = \Omega(u, v)e^{iS(u,v)}$. When the quantum potential is zero, the identification

$$\frac{\partial S}{\partial q_i} = p_i = \frac{\partial}{\partial \dot{q}_i} \tag{10}$$

is possible. If this classical definition for the momenta is retained even when $Q \neq 0$, the semiclassical solutions will differ from the classical ones.

Under the assumption that the quantum corrections in the general solution (7) follow from the "frequency $I_0$" with the highest peak in the wave function, which is in agreement with the so-called Hartle criterion[14], the quantum potential vanishes. This provides no quantum corrections and the semiclassical equations (10) give the classical solution. This is an indication that the Szekeres universe remains "silent", even at the quantum level.

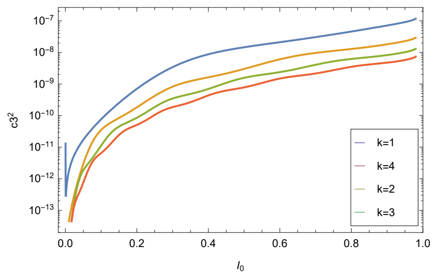


Fig. 1. Qualitative evolution of the normalize parameter $c_3^2$ in terms of the free parameter $I_0$ for $k = 1$ (blue line), $k = 2$, (yellow line), $k = 3$ (green line) and $k = 4$ (red line). From the plot we observe that $c_3^2$ goes to zero for values of $I_0$ close to zero.

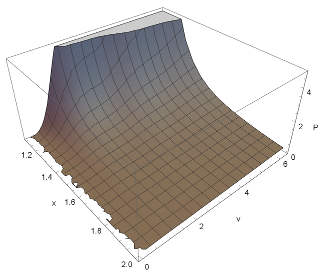


Fig. 2. Qualitative evolution of the probability function in the space of variables $x, v$.

In the case $h = 0$ and $\Psi_1 \to 0$, the wave function is well behaved at $u \to 0$ and $u \to \infty$. We can thus define a probability which, after a change of coordinates to $u \to \frac{2}{x^2 - I_0}$ becomes

$$P = \int_{\sqrt{I_0}+\epsilon}^{\lambda} dx \int_0^{2k\pi} dv \; \frac{4c_3^2 \sin(xv)}{x(x^2 - I_0)^2} \; , \; k \in \mathbb{N}. \tag{11}$$

where the cut-off constant $\lambda$ is introduced to exclude the case $E = 0, \rho = 0$. The normalization gives a quantized value for the constant $c_3$. Its qualitative evolution is given in Fig. 1. The qualitative behaviour of the probability function is given in the surface diagram in Fig. 2 and the contour plot in Fig. 3. The plots show that for $I_0 \to 0$ the probability function reaches its minimum.

## 4. Conclusions

Our quantum analysis of the Szekeres system was based on the canonical quantization in the presence of symmetries and the results were interpreted by adopting the Bohmian mechanics approach. The starting point was an effective classical point-like Lagrangian which can reproduce the two dimensional system of second-order differential equations resulted from the initial field equations. This Lagrangian is autonomous, thus there exists a conservation law of "energy" corresponding to the

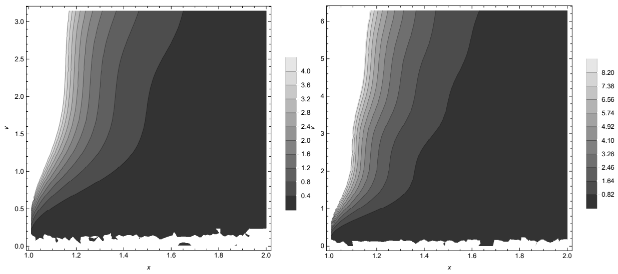

Fig. 3. Contour plot for the probability function in the space of variables $x, v$. We observe that as $x \to 0$ and $v$ is small, that is, $I_0 \to 0$, the function $P(x, v)$ reaches to a minimum extreme.

Hamiltonian function. As for the extra contact symmetry, it leads to a quadratic in the momenta conserved quantity attributed to a Killing tensor of the second-rank. The two conserved quantities give two eigenequations at the quantum level, the Hamiltonian function being the Schrödinger equation.

The assumption that the wave function is peaked around its classical value. This leads to the lack of quantum corrections and the recovery of the classical solutions, thus leading to the conclusion that the Szekeres universe remains silent at the quantum level. Finally, for the particular case $h = 0$ it was shown that the probability function and relate one (unstable) exact solution with the existence of a minimum of this probability.

At this point, we would like to remind that the Szekeres system admits the exact solution $u_A(t) = \frac{6^{\frac{2}{3}}}{2} t^{\frac{2}{3}}, v_A(t) = v_0 t^{-\frac{1}{3}}$, in which the integration constants $h$ and $I_0$ are zero[9]. The latter solution corresponds to an unstable critical point for the dynamical system (2) and it is very interesting that the conditions for the existence of the exact solution, i.e. $h = 0$ and $I_0 = 0$, lead to an extremum for the probability function. This might be related with the existence and stability of the exact solution. The fact that the quantum probability has its minimum at the classical value is in accordance with the analysis of the probability extrema in[15] where it was shown that the extrema of the probability lie on the classical values.

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