The non perturbative gyro-phase is the *Kaluza-Klein* 5^{th} dimension

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The curve in space occupied by the mass during time evolution, is a geodesic on the space-time manifold curved by the presence of masses: the mass can only follow its trajectory consistently with the underlying gravitational field. Is it possible to think at the charge trajectory in a similar fashion? Is it possible to say that the charge trajectory, the curve in phase-space occupied by the charge during time evolution, is the geodesic on the extended phase-space curved by the presence of charges? If yes then it should be possible to obtain an Einstein's equation also for electromagnetism. This is done by considering a metric on the whole extended phase-space. It is proposed to add a Hilbert-Einstein term in the lagrangian when velocities are considered as dynamical variables. Here, it will be analyzed what happens if the (non perturbative) guiding center description of motion is adopted [1]. In such case, a similar mechanism to the one proposed by Kaluza and Klein (KK) [2,3] a century ago is found. The advantage of using the present description is that, now, there is no need of looking for a compactification scheme as required in the original KK mechanism. Indeed, the extra-dimension that appears in the guiding center transformation is a physical and, in principle, measurable variable being the gyro-phase, the angle obtained when the velocity space is described in a sort of cylindrical transformation of velocities coordinates. Regardless of the equations that are really similar to the one seen in the KK mechanism, the new claim is in the interpretation of the extra dimension as a coordinate coming from the phase-space. Until now, all the compactification mechanisms have been shown to give problems, like the inconsistency of the scale of masses with observations. Instead, without a compactification at the Planck scale length and giving a physical meaning to the extra-coordinate, it seems that the KK mechanism can finally be accepted as a realistic explanation of the presence of gravitation and electromagnetism treated in a unified manner in general relativity theory extended to higher dimensions.

Keywords: Kaluza Klein, Guiding Center Transformation,
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1. Introduction

The thesis of the present work is that through the guiding center transformation [1,4] it is possible to support the Kaluza-Klein model [2,3,5,6] without the need to use new physical dimensions with respect to those of the phase space extended to time. What turns out is that the extra dimension comes from the velocity space. There are two constants of motion, the single particle lagrangian that does not change value with respect to the guiding center transformation and the magnetic moment that does not change value with respect to the variation of the gyrophase. In the non perturbative guiding center transformation it is assumed that there exist a reference point, the guiding center, from where the particle motion is seen to be closed and periodic. Thus the motion of a charged particle is represented by the product of the guiding center orbit times the circle with gyroradius, ρ. Such helicoid motion is implicitly written in a flat phase space as

$$x = X + \rho(t, X, \gamma; \mu, \varepsilon) \tag{1}$$
$$u = U(t, X; \mu, \varepsilon) + \nu(t, X, \gamma; \mu, \varepsilon),$$

where $u = x'$ is the relativistic velocity of the particle, $U = X'$ is the relativistic velocity of the guiding center and $\nu = \rho'$ is the difference or the proper time derivative of the gyroradius. The relativistic lagrangian is:

$$L = -p_\alpha u^\alpha, \tag{2}$$

being the co-momentum $p_\alpha = u_\alpha + (e/m)A_\alpha(x^\beta)$, with A_α the e.m. potential. Also the guiding center has a co-momentum defined as $P_\alpha = U_\alpha + (e/m)A_\alpha(X^\beta)$, in such a way that the lagrangian in (2) is

$$L = -p_\alpha u^\alpha = -P_\alpha U^\alpha - (e/m)g'. \tag{3}$$

If g is proportional to the product of the magnetic moment, μ, times the gyro-phase, γ, it is called the *guiding center gauge function* : $g = (m/e)\mu\gamma$. Such gauge function doesn't alter the equation of motion but it is useful for preserving the same value of the lagrangian along the motion. The link with the KK model starts from denoting the values $z^0 = t$, $z = X$, $z^4 = \gamma$ and $w_0 = P_0 = U_0 + (e/m)\Phi$, $w = P = U + (e/m)A$ and $w_4 = (m/e)\mu$, then

$$L = -w_a z'^a, \qquad \text{for a=0,1,2,3,4.} \tag{4}$$

which is a scalar product in a space-time of five dimensions. Moreover, if you require that $w_5 = w_6 = 0$ and $z^5 = \mu$, $z^6 = \varepsilon$ is the particle energy (per unitary mass), then you can also write

$$L = -w_A z'^A, \qquad \text{for A=0,1,2,3,4,6.} \tag{5}$$

The latter is what is called the phase-space lagrangian from which it is possible to find the *Hamilton's equations*. It is better to denote with a *hat* the guiding center phase-space lagrangian: $L = \hat{L}(z^A, z'^B)$, for $A, B = 0, 1, 2, 3, 4, 6$. As said in[7], the reason for the vanishing of w_5 and w_6 is due to the fact that ε is the conjugate coordinate of t and $(m/e)\mu$ is the conjugate coordinate of γ.
Now, the lagrangian is invariant at a glance with respect to general non-canonical phase-space coordinates transformations, that include also the gauge transformations.

2. Kaluza-Klein solution

The coordinates z^A with $A = 0, 1, 2, 3, 4, 5, 6$, just introduced, belong to the extended phase space. As for general relativity, where a geometry is given to the space-time, in this section a geometry is given to the extended phase-space.

Let's start from the *Poincaré-Cartan* one-form from (5): $\hat{L}d\hat{s} = -w_A dz^A$, for $A = 0, 1, 2, 3, 4, 5, 6$. The same one-form can be written as

$$\hat{L}d\hat{s} = -\hat{g}_{AB} w^B dz^A, \tag{6}$$

being $\hat{L}$ a scalar quantity and where $\hat{g}_{AB}$ is the metric tensor with the property that $w_A \equiv \hat{g}_{AB} w^B$. Thus, w^B are the *contravariant* momenta. Once the metric tensor is appeared, it is possible to apply a variational principle for finding it. For this reason, we consider a *lagrangian density over the extended phase space* where the single particle lagrangian is multiplied for the distribution of masses and, then, added to the HE lagrangian in extended dimensions. In the following the lagrangian *distribution* is

$$\ell a = f_m \hat{L} - \frac{\hat{\mathcal{R}}}{16\pi\hat{G}}, \tag{7}$$

Thus, the action is:

$$S = \int \ell a \, d\mathcal{M}, \tag{8}$$

which is a definite integration in a domain $\partial\mathcal{M}$ of the extended phase space. It is possible to separate in ℓa the effects of different contributions: a *matter*, a *field* and an *interaction lagrangian distribution*. Concerning the *field action*, S_{f}, we have:

$$S_{\mathrm{f}} = -\int \frac{\hat{\mathcal{R}}}{16\pi\hat{G}} \sqrt{|\hat{g}|} d^7 z = \tag{9}$$
$$= -\int \frac{F_{\alpha\beta}F^{\alpha\beta}}{4} \sqrt{-g} dt d^3 X - \int \frac{R}{16\pi G} \sqrt{-g} dt d^3 X.$$

In order to obtain the latter result the KK mechanism is applied. At the same time, the action due to the interaction:

$$S_{\mathrm{id}} = \int f_m (1 + \hat{L}) \sqrt{|\hat{g}|} d^7 z = -\int A_\alpha J^\alpha \sqrt{-g} dt d^3 X, \tag{10}$$

where J^α is the charge four-current density which is a field depending on (t, X). The former equation is obtained through the non perturbative guiding center transformation and the misleading symmetry, below described. It is worth noticing that once integrated in the velocity space, the obtained lagrangian density is the one used for describing the presence of matter as source of a gravitational field, which gives the *Einstein* equation, together with a charge four-current density as source of an e.m. field, which gives the *Maxwell* equations. It is worth noticing that after the integration over the velocity space the model equations have lost locality.

2.1. *The* misleading *symmetry*

In the relativistic case, it is chosen to preserve the product $u^\alpha A_\alpha(x^\beta) = U^\alpha A_\alpha(X^\beta)$ that allows to write $L = -1 + (e/m)u^\alpha A_\alpha(x^\beta) = \hat{L}$ with

$$\hat{L} = -1 + (e/m) U^\alpha A_\alpha(X^\beta), \tag{11}$$

which is the same form of L. The required condition is reached if

$$(m/e)\mu\gamma' = 1 - U^\alpha U_\alpha. \tag{12}$$

The latter relation is also more interesting if $(m/e)\mu\gamma' = U^4 U_4$, where $U^4 = z^{4\prime} = \gamma'$ and U_4 is firstly defined as $U_4 \equiv w_4 = (m/e)\mu$. In such way that

$$U^a U_a = 1, \qquad \text{for } a = 0, 1, 2, 3, 4. \tag{13}$$

Moreover, if the relation $w_a = U_a + (e/m)A_a$ is used, then $A_4 = 0$ for consistency: there is not a 5^{th} component of the e.m. potential. The symmetry that leaves invariant the form of $L = -1 + (e/m)u^\alpha A_\alpha(x^\beta) = -1 + (e/m)U^\alpha A_\alpha(X^\beta)$ is said *misleading* because there is no way, starting from the dynamics, *e.g.* from the lagrangian, to distinguish particle's coordinates from guiding center's coordinates. The only chance for appreciating the difference is by measuring the dispersion relation: from kinematics, the particle has $u^\alpha u_\alpha = 1$ whilst the guiding center doesn't, $U^\alpha U_\alpha \neq 1$, that means that the guiding center is a virtual particle.

3. Conclusion

We have just seen that the guiding center transformation, which is a particular *local* translation in the extended phase space, *e.g.* see (1), is a symmetry because it leaves the same lagrangian form. In analogy to what happens for the *local* translation in spacetime, the conserved quantity for the present symmetry should be called the *extended energy-momentum* tensor $\hat{T}_{AB}$. Now, the *Einstein tensor* for the extended phase space is obtained from the variation of $-\hat{\mathcal{R}}/16\pi\hat{G}$ with respcet to $\delta\hat{g}^{AB}$:

$$\hat{G}_{AB} = \hat{Ric}_{AB} - \hat{R}\,\hat{g}_{AB}/2, \tag{14}$$

and the *Einstein equation* can be written also for the extended phase space,

$$\hat{G}_{AB} = 8\pi\hat{G}\,\hat{T}_{AB}. \tag{15}$$

It is worth noticing that, if confirmed, we have just obtained gravitation and electromagnetism from a geometrical perspective.

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