A geometric relativistic dynamics under any conservative force

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Abstract

Riemann's principle "force equals geometry" provided the basis for Einstein's General Relativity - the geometric theory of gravitation. We introduced a Generalized Principle of Inertia stating that: "An inanimate object moves inertially (with constant velocity) in its own spacetime whose geometry is determined by the forces affecting it". Classical Newtonian dynamics for motion under a conservative force field is treated within this framework, using a properly defined Newtonian metric on an inertial lab frame which is obtained from the potential of the force field acting on the object. We reveal a physical deficiency of this metric (responsible for the inability of Newtonian dynamics to account for relativistic behavior), and remove it. The dynamics defined by the corrected Newtonian metric leads to a new Relativistic Newtonian Dynamics for both massive objects and massless particles moving in any static, conservative force field, not necessarily gravitational. This dynamics reduces in the weak field, low velocity limit to classical Newtonian dynamics and also exactly reproduces the tests of General Relativity.

I. SUMMARY

Bernhard Riemann, although best known as a mathematician, became interested in physics in his early twenties. His lifelong dream was to develop the mathematics to unify the laws of electricity, magnetism, light and gravitation. Riemann's approach to physics was geometric. As pointed out in [1], "one of the main features of the local geometry conceived by Riemann is that it is well suited to the study of gravity and more general fields in physics." He believed that the forces at play in a system determine the geometry of the system. For Riemann, *force equals geometry*.

The application of Riemann's mathematics to physics would have to wait for two more essential ideas. While Riemann considered how forces affect *space*, physics must be carried out in *space-time*. One must consider trajectories in spacetime, not in space. For example, in flat spacetime, an object moves with constant velocity if and only if his trajectory in *spacetime* is a straight line. On the other hand, knowing that an object moves along a straight line in *space* tells one nothing about whether the object is accelerating. As Minkowski said, "Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality [2]." This led to the second idea. Riemann worked only with positive definite metrics, whereas Minkowski's metric on spacetime is not positive definite. The relaxing of the requirement of positive-definiteness to non-degeneracy led to the development of *pseudo-Riemannian geometry*.

GR is a direct application of "force equals geometry." In *GR*, the gravitational force curves spacetime. Since, by the Equivalence Principle, the acceleration of an object in a gravitational field is independent of its mass, curved spacetime can be considered a *stage* on which objects move. In other words, the geometry is the same for all objects. However, the Equivalence Principle holds only for gravitation. In this way, *GR* singles out the gravitational force from other forces which are not treated geometrically. For example, the potential of an electric force depends on the charge of the particle, and the particle's acceleration depends on its charge-to-mass ratio. Thus, the electric field does not create a common stage on which all particles move. Indeed, a neutral particle does not feel any electric force at all. The way spacetime curves due to an electric potential depends on *both* the potential *and* intrinsic properties of the object. How, then, are we to apply Riemann's principle of "force equals geometry" to other forces?

In this talk, we realize Riemann's program for motion in any static, conservative force field.

One of the main new ideas is the *relativity of spacetime*. By this, we mean that spacetime is an object-dependent notion. An object lives in its own spacetime, its own geometric world, which is defined by the forces which affect it. For example, in the vicinity of an electric field, a charged particle and a neutral particle exist in different worlds, in different spacetimes. Likewise, in the vicinity of a magnet, a piece of iron and a piece of plastic live in two different worlds.

An inanimate object has no internal mechanism with which to change its velocity. Hence, it has constant velocity in *its own* world (spacetime). This leads us to formulate a new principle, the *Generalized Principle of Inertia*, which generalizes Newton's First Law and states that: **An inanimate object moves inertially, that is, with constant velocity, in *its own* spacetime whose geometry is determined by the forces affecting it**. This is a generalization, or more accurately, a relativization of what Einstein accomplished. In *GR*, an object freely falling in a gravitational field is in free motion. This is attested to by the fact that along a geodesic, the acceleration is zero. The Generalized Principle of Inertia states that *every* object is in free motion in *its* spacetime, determined by the forces affecting it. Since, by the Generalized Principle of Inertia, an object moves with constant velocity in its own spacetime, we assume that there exists a metric with respect to which the length of the object's trajectory is extremal. This metric, which we call the *metric of the object's spacetime,* will depend only on the forces, and, in the case of static, conservative forces, the metric will depend only on the combined potential of these forces..

Since the object's metric extremizes the length of trajectories, its worldline is derived from the conservation rules resulting from a variational principle. Let $q : \sigma \rightarrow x, a \leq \sigma \leq b$ be a trajectory of an object, where σ is an arbitrary parameter. Let $ds^2 = g_{ij}(q)dq^i dq^j$ be the metric of the object's spacetime. Define

$$L(q, \acute{q}) = \frac{ds}{d\sigma} = \sqrt{g_{ij}(q)\acute{q}^i \acute{q}^j}, \tag{1}$$

where $\acute{q} = \frac{dq}{d\sigma}$. The length $l(q)$ of the trajectory q does not depend on the parametrization and is given by $\int_a^b \frac{ds}{d\sigma} d\sigma = \int_a^b L(q, \acute{q}) d\sigma$.

We show that classical Newtonian dynamics for motion under a static, conservative force with potential U extremizes distances under the metric

$$ds^2 = (1 - u(\mathbf{x}))c^2 dt^2 - \frac{1}{1 - u(\mathbf{x})}d\mathbf{x}^2, \tag{2}$$

where $u(\mathbf{x}) = \frac{-2U(\mathbf{x})}{mc^2}$ denotes the dimensionless potential. The huge success of Newtonian dynamics implies that the Newtonian metric (2) is close to the one that governs the laws of Nature.

Nevertheless, the observed astrophysical deviations from the predictions of this dynamics indicate that this metric has a deficiency and needs to be corrected.

The metric (2) is deficient in that it is isotropic - it alters the spatial increments equally in *all* spatial directions. The potential, on the other hand, influences only the direction of the gradient ∇U and has no influence on the spatial increments in the directions transverse to the gradient. To remove this problem, we introduce at each $\mathbf{x}$ a normalized vector $\mathbf{n}(\mathbf{x}) = \nabla U(\mathbf{x})/|\nabla U(\mathbf{x})|$ in the direction of the gradient of $U(\mathbf{x})$ and denote by $d\mathbf{x}_n$ and $d\mathbf{x}_{tr}$, respectively, the projections of the spatial increment $d\mathbf{x}$ in the parallel and transverse directions to $\mathbf{n}(\mathbf{x})$. With this notation, the *corrected Newtonian metric* is

$$ds^2 = (1 - u(\mathbf{x}))c^2 dt^2 - \frac{1}{1 - u(\mathbf{x})}d\mathbf{x}_n^2 - d\mathbf{x}_{tr}^2. \tag{3}$$

We will call the dynamics resulting from this metric *Relativistic Newtonian Dynamics* (*RND*).

In the case of the gravitational field of a non-rotating, spherically symmetric body of mass M, in spherical coordinates with origin at its center, the potential is $U(r) = -GmM/r$, and the dimensionless potential is $u(r) = 2GM/c^2 r = r_s/r$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. In this case, the metric (3) is

$$ds^2 = \left(1 - \frac{r_s}{r}\right)c^2 dt^2 - \frac{1}{1 - r_s/r}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{4}$$

which is the well-known Schwarzschild metric ([3]). This implies that *RND* reproduces the tests of *GR*, as shown in [4–7].

The *RND* energy has, in addition to the usual kinetic and a potential energies term, a *mixed term* which depends on both the velocity of the object and the potential. This means that in order to reproduce relativistic effects, one can no longer distinguish between potential and kinetic energy, as in Newtonian dynamics. This also explains the need to include the velocity in the modified Newtonian potentials proposed in [8–13].

The *RND* dynamics equation has adds two new terms to Newton's second law and reduces to it in the low velocity, weak field limit. Kepler's laws of planetary motion in celestial mechanics provided the basis for Newtonian physics, applicable until today to all forces of Nature in the non-relativistic regime. In a similar way, we expect *RND* to provide the basis for relativistic physics

for other forces of Nature.

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