Strong Quantum Energy Inequality and the Hawking Singularity Theorem

Eleni-Alexandra Kontou

Department of Mathematics, University of York

Abstract

Hawking's singularity theorem concerns matter obeying the strong energy condition (SEC), which means that all observers experience a non-negative effective energy density (EED), thereby guaranteeing the timelike convergence property. However, for both classical and quantum fields, violations of the SEC can be observed in some of the simplest of cases, like the Klein-Gordon field. Therefore there is a need to develop theorems with weaker restrictions, namely energy conditions averaged over an entire geodesic and quantum inequalities, weighted local averages of energy densities. We have derived lower bounds of the EED in the presence of both classical and quantum scalar fields allowing nonzero mass and nonminimal coupling to the scalar curvature. In the quantum case these bounds take the form of a set of state-dependent quantum energy inequalities valid for the class of Hadamard states. Finally, we discuss how these lower bounds are applied to prove Hawking-type singularity theorems asserting that, along with sufficient initial contraction at a compact Cauchy surface, the spacetime is future timelike geodesically incomplete. The talk is based on arXiv:1803.11094 and a manuscript in preparation.

A spacetime is defined to be singular if it possesses at least one incomplete geodesic. The question of whether or not cosmological models either originate or terminate in singularities has been an active subject of research since the formulation of the general theory of relativity. Initial efforts focused on models with high levels of symmetry, with the first decisive step forward in 1955 with the work of Raychaudhuri. The Raychaudhuri equations in their modern form [3] present the evolution of geodesic congruences, and for the case of an timelike irrotational congruence with velocity field U^μ, the expansion $\theta = \nabla_\mu U^\mu$ satisfies

$$\nabla_U \theta = R_{\mu\nu} U^\mu U^\nu - 2\sigma^2 - \frac{\theta^2}{n-1}\,, \tag{1}$$

where n is the spacetime dimension, σ is the shear scalar and $R_{\mu\nu}$ is the Ricci tensor. The Raychaudhuri equations are the heart of all singularity theorems. Senovilla [10] has described the skeleton of the singularity theorems in terms of a 'pattern theorem' with three ingredients. An energy condition establishes a focussing effect for geodesics, while a causality condition removes the possibility of closed timelike curves, and a boundary or initial condition establishes the existence of some trapped region of spacetime. We will divide singularity theorems into 'Hawking-type' from Hawking's original theorem [6] and 'Penrose-type' from Penrose's theorem [9], depending on whether they demonstrate timelike or null geodesic incompleteness respectively. Hawking-type results which concern us here are based on the strong energy condition (SEC)

$$\rho = T_{\mu\nu}U^\mu U^\nu - \frac{T}{n-2} \geq 0\,, \tag{2}$$

which requires that the effective energy density ρ is positive.

For the minimally coupled Klein-Gordon field the EED is

$$\rho = (\nabla_U \phi)^2 - \frac{m^2\phi^2}{n-2}, \tag{3}$$

which is easily made negative at individual points. This situation is exacerbated in quantum field theory, in which none of the pointwise energy conditions can hold [4].

For these reasons there has long been interest in establishing singularity theorems under weakened energy assumptions given averaged energy conditions that bound the weighted energy density along an entire geodesic

$$\int_\gamma d\tau \rho\, f^2(\tau) \geq 0\,, \tag{4}$$

and quantum inequalities that introduce a restriction on the possible magnitude and duration of any negative energy densities or fluxes within a quantum field theory

$$\int d\tau f^2(\tau) \langle \rho^{\text{quant}} \rangle_\Psi (\gamma(\tau)) \geq -A\,. \tag{5}$$

Examples in which various averages of the energy density are required to be nonnegative include [11, 2, 1]. Our approach in this work follows [5], which showed that suitable lower bounds on local weighted averages of ρ are sufficient to derive singularity theorems of both Hawking and Penrose types, even if ρ is not positive everywhere or has a negative long-term average.

In this talk I will discuss two results in the direction of weakened energy assumptions of Hawking-type singularity theorems:

- A singularity theorem obeyed by the classical non-minimally coupled Einstein-Klein-Gordon field .
- The derivation of a difference strong quantum energy inequality for the non-minimally coupled quantum scalar field, and the proof of a singularity theorem with an energy condition obeyed by the minimally coupled quantum scalar field.

First I will sketch the derivation of a bound for the EED of the non-minimally coupled Klein-Gordon field integrated over an entire timelike geodesic. Next I will show how this bound can be adapted to provide more specific information about solutions to the full Einstein-Klein-Gordon system. This bound allows us to prove a Hawking-type singularity theorem: For (M, g, ϕ) a solution to the Einstein-Klein-Gordon equation with coupling less than the conformal coupling, (M, g) admitting a smooth spacelike Cauchy surface, ϕ obeying global bounds and enough initial contraction, (M, g) is future geodesically incomplete. This shows that sufficient initial contraction on a compact Cauchy surface is enough to guarantee timelike geodesic incompleteness, even though the non-minimally coupled Klein-Gordon theory does not obey the SEC. I will also discuss the hypotheses of the singularity theorem in more depth, and estimate the physical magnitude of the initial contraction required. I will show that a model in which the field mass is taken equal to an elementary particle mass would need very little initial contraction to guarantee that either there is geodesic incompleteness or that at least, the solution evolves to a situation where the natural energy scales associated with the field approach those of the early universe.

The second section begins with a brief discussion of the quantization of EED. We use the algebraic approach to quantum field theory where objects $\Phi(f)$ are members of unital *-algebra $\mathscr{A}(M)$ on our manifold M and obey a set of relations. We only consider Hadamard states on our algebra where the two-point function ω_2^Ψ has a prescribed singularity structure, so that the difference between two states is smooth. Next we need a prescription for finding algebra elements that qualify as local and covariant Wick powers. This can be done in various ways, expressing finite renormalisation freedoms. Hollands and Wald [7, 8] set out a list of axioms that we follow. Utilizing these definitions, any classical expression constructed from the stress energy tensor such as the EED of the non-minimally coupled scalar field

$$\rho(x) = [\hat{\rho}(\phi \otimes \phi)]_c(x)\,, \tag{6}$$

has a quantized form in Hadamard state Ψ defined by

$$\langle \rho(x) \rangle_\Psi(x) = [\hat{\rho}\colon \omega_2^\Psi \colon]_c(x)\,. \tag{7}$$

Here we used $:\omega_2^\Psi: \; = \omega_2^\Psi - \omega_2^0$, the normal ordering of the two-point function.

Using techniques like point-splitting, we derive a non-trivial bound for the weighted average of the quantized EED over an entire timelike geodesic. If we constrain the state ψ and the metric $g_{\mu\nu}$ to those that satisfy the semiclassical Einstein equation

$$\langle : T_{\mu\nu} : \rangle_\Psi = 8\pi G_{\mu\nu}\,, \tag{8}$$

and consider only minimally coupled fields we derive an inequality with only geometric terms on the left hand side, thus appropriate as a condition for a singularity theorem. A final obstacle is the fact that the bound depends on the reference state but in curved spacetimes there is no preferred vacuum. However, restricting the sampling function f to have limited support we can argue that in small scales compared to curvature, there is a vacuum state that corresponds to Minkowski vacuum. Using a partition of unity we can decompose the integral, obtaining a sum of integrals, each of which can be bounded. The final bound is similar to classical one and it can be used as a condition to a Hawking-type singularity theorem considering the appropriate initial contraction.

As the classical singularity theorems have in their hypotheses easily violated energy conditions there is an ongoing effort to prove ones with weakened conditions, obeyed by known matter. In this talk I will present the derivation of a Hawking-type singularity theorem with an energy condition obeyed by the classical non-minimally coupled Einstein-Klein-Gordon field and one with an energy condition derived by a QEI obeyed by the minimally coupled quantum scalar field.

References

[1] Arvind Borde. Geodesic focusing, energy conditions and singularities. *Classical Quantum Gravity*, 4(2):343–356, 1987.

[2] Carmen Chicone and Paul Ehrlich. Line integration of Ricci curvature and conjugate points in Lorentzian and Riemannian manifolds. *Manuscripta Math.*, 31(1-3):297–316, 1980.

[3] Jürgen Ehlers. Contributions to the relativistic mechanics of continuous media. *Gen. Relativity Gravitation*, 25(12):1225–1266, 1993. Translation from German of an article originally published in Akad. Wiss. Lit. Mainz Abh. Math.-Nat. Kl., Nr. 11, 792–837 (1961).

[4] H. Epstein, V. Glaser, and A. Jaffe. Nonpositivity of energy density in Quantized field theories. *Nuovo Cim.*, 36:1016, 1965.

[5] Christopher J. Fewster and Gregory J. Galloway. Singularity theorems from weakened energy conditions. *Class. Quant. Grav.*, 28:125009, 2011.

[6] Stephen Hawking. The Occurrence of singularities in cosmology. *Proc. Roy. Soc. Lond.*, A294:511–521, 1966.

[7] Stefan Hollands and Robert M. Wald. Local Wick polynomials and time ordered products of quantum fields in curved space-time. *Commun. Math. Phys.*, 223:289–326, 2001.

[8] Stefan Hollands and Robert M. Wald. Conservation of the stress tensor in interacting quantum field theory in curved spacetimes. *Rev. Math. Phys.*, 17:227–312, 2005.

[9] Roger Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, 1965.

[10] José M. M. Senovilla. Singularity theorems and their consequences. *Gen. Relativity Gravitation*, 30(5):701–848, 1998.

[11] Frank J. Tipler. Energy conditions and spacetime singularities. *Phys. Rev.*, D17:2521–2528, 1978.