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Curvature and Torsion in Quantum Geometrodynamics

Aalok*

Department of Physics, University of Rajasthan, Jaipur 302004 India; and
Department of Physics, JECRC University, Jaipur 303905 India

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Formulation of curvature and torsion in Quantum Geometrodynamics is discussed. Torsion in quantum evolution for symmetric states is found to be $\tau^2 = \frac{\langle H^6 \rangle}{\langle H^4 \rangle \langle H^2 \rangle}$. Curvature in the quantum evolution formulated earlier by Brody and Houghston in terms of moments of Hamiltonian as $\kappa^2 = \frac{\langle H^4 \rangle}{\langle H^2 \rangle^2} - \frac{\langle H^3 \rangle^2}{\langle H^2 \rangle^3}$ is verified. Thus, the formulation of curvature in quantum evolution is reassuring in more than one ways. The Geometry of $Serret - Frenet$ formulae is recast in the context of Geometric Quantum Mechanics. The Geometry of quantum neighborhood also leads to the formulation of curvature and torsion during quantum evolution. Estimator problem when subjected to neighborhood test brings about many significant results. Fourth order term in the quantum neighborhood test carries information of curvature whereas the sixth order term conveys significant information regarding torsion.

I. INTRODUCTION

The discussion in this paper is inspired by the formulation of curvature in Quantum Mechanics by Brody and Houghston [1] and [2]. In the follow up we propose to formulate torsion in Quantum Mechanics and reproduce the expression of curvature which in turn strengthens the foundations of Geometric Quantum Mechanics. The premise of Geometric Quantum Mechanics is woven around the notion of neighbourhood. The notions of metric, distance and geometric phase in quantum evolution are all about nighbourhood in the topological sense. Exercises in the present discussion too employ the same techniques and the premise to explore curvature and torsion in quantum evolution. At the outset, we briefly review the background and the perspective in which the present exercise has been carried out. As a quantum system evolves in time the state vector changes and it traces out a curve in the Hibert space $\mathscr{H}$. Geometrically, the evolution is represented as a closed curve in the projective Hilbert space $\mathscr{P}(\mathscr{H})$ [3-9]. The idea of representing quantum state space by complex projective space corresponding to finite dimensional Hilbert space is now on firm footing [3-7, 10-14]. The distance on the projective Hilbert space is defined in terms of metric, called the metric of the ray space that is projective Hilbert space $\mathscr{P}$ and can be regarded as an alternative definition of the Fubini-Study metric valid for a finite dimensional Hilbert space $\mathscr{H}$. The metric in the ray space is now being referred by physicists as the Background Independent and space-time independent structure can play an important role in the construction of a potential theory of quantum gravity. In the light of recent studies [2, 3, 10-17] of geometry of the quantum state space the extension of standard geometric Quantum Mechanics is irresistible. Researchers studying gravity have also shown considerable interest in the geometric structures in Quantum Mechanics in general and projective Hilbert space in specific [1-3, 14-17]. Thus, an intensive follow up will be academically rewarding [3]. There is prevailing spirit that one can recast Quantum Mechanics in a geometric language which brings about all the niceties that are relevant to these studies [3]. The author feels that the discussion in the present paper will turn out be of substantial interest to these studies by means of studies of Background Independent Quantum Mechanics (BIQM). The idea of geometrization of Quantum Mechanics is to move from Hilbert space to the space of rays which is the 'true' space of states of Quantum Mechanics that is projective Hilbert space $\mathscr{P}(\mathscr{H})$ and the corresponding manifold is Kähler structure. The probabilistic interpretation of Quantum Mechanics is thus inherent in the metric properties of $\mathscr{P}(\mathscr{H})$ [14-17]. The symmetries of the Geometry of CP^N are prescribed by the quotient set

$$CP^N \equiv \frac{U(N+1)}{U(N) \times U(1)}. \tag{1}$$

Obviously, it has its limitations as it is valid for finite dimensional Hilbert space only. Thus, the only alternative that seem to satisfy almost complex structure is the *Grassmanian*. By the correspondence principle, the genelized quantum geometry must locally recover the canonical quantum theory encapsulated in $P(N)$ and also allows for

*Electronic address: aalok.pandya@jecrcu.edu.in; URL: http://www.jecrcuniversity.edu.in/faculty/profile/dr-aalok

mutually compatible metric and symplectic structure, and supply the framework for dynamical extension of the canonical quantum theory. The *Grassmanian*

$$Gr(C^{N+1}) = \frac{Diff(C^{N+1})}{Diff(C^{N+1}, C^N \times 0)}. \tag{2}$$

II. CLASSICAL GEOMETRY $VIS - A - VIS$ GEOMETRIC QUANTUM MECHANICS: AN OVERVIEW

We briefly discuss the essentials of geometry of Hilbert space, the geometry of quantum state space and the tangent space in order to build further geometric structures.

Evolution equation and tangent bundle

We first analyze the equation of quantum evolution that is Schrödinger's equation and then discuss the underlying geometry. We consider point $p \in P(N)$, corresponding to a quantum state $\mid \Psi\rangle$ that is a point at which the quantum state could be projected into the state space. In a given representation [11-14], $\mid \Psi\rangle$ can be expressed as a column matrix $(\alpha^0, \alpha^1, \alpha^2, ...)^T$, and the matrix of the Schrödinger's equation goes as:

$$i\hbar \frac{d\alpha_i(t)}{dt} = \Sigma_j H_{ij} \alpha^j. \tag{3}$$

We can define tangent vector on CP^N since it is a complex differential manifold [11-14]. We choose the local coordinates of the point $p \in P(N)$ as $(\alpha^0, \alpha^1, \alpha^2, ...)$, then the tangent vector on the point p is

$$\mathbf{T} = \frac{d\alpha_i}{dt} \frac{\partial \Psi}{\partial \alpha_i}; \tag{4}$$

where t is an arbitrary parameter. In a dynamical scenario t could be time as well, and $\frac{d\alpha_i}{dt}$ determines the components of $\mathbf{T}$ when we choose our local coordinates $(\alpha^0, \alpha^1, \alpha^2, ...)$ on CP^N.

III. THE GEOMETRY OF $SERRET - FRENET'S$ FORMULAE IN QUANTUM EVOLUTION

We can recast the geometry of $Serret - Frenet$ formulae on the quantum state space too. It is worth mentioning here that curvature can be formulated by means of evaluating lower bounds on the variance of the estimator as was done by Brody and Houghston [1]. However, we choose to deduce expressions of curvature and torsion following first principles of Differential Geometry.
One can define tangent vector on CP^N as discussed earlier in this paper and in the references [10-17]. The Geometry of $Serret - Frenet$ formulae can be discussed alternatively in terms of the *arc* parameter s on the geodesic that is Fubini-Study metric

$$ds^2 = \Delta E^2 dt^2 = [\langle \partial_i \Psi \mid \partial_j \Psi\rangle - \langle \partial_i \Psi \mid \Psi\rangle \langle \Psi \mid \partial_j \Psi\rangle] d\alpha^i d\alpha^j. \tag{5}$$

This is geodesic distance between any two quantum states $\mid \Psi_1\rangle$ and $\mid \Psi_2\rangle$ and is known as metric of quantum states on the projective Hilbert space which is the space of quantum states. Also, it should be noticed that the invariant in the eq. (8) is independent of choice of parameter α_i. The length of a geodesic connecting two quantum states $\mid \Psi_1\rangle$ and $\mid \Psi_2\rangle$ is also referred to as Wooter's distance by the following expression:

$$s = \int ds = \gamma[arccos(\mid \langle \Psi_1 \mid \Psi_2\rangle \mid)]. \tag{6}$$

Where, a tangent at a point $p \in CP^N$ on the base manifold of quantum states is defined as:

$$\mathbf{T} = \frac{\partial \Psi}{\partial s} \frac{ds}{dt}; \tag{7}$$

where, the tangent lies on the tangent space $T_p(CP^N)$ and

$$\frac{ds}{dt} (= v) = \langle (\Delta E)^2 \rangle^{\frac{1}{2}}; \tag{8}$$

can be called velocity of quantum evolution. It is also possible to define normal and bi-normal as function of local coordinates on the quantum state space using expression of tangent: $\mathbf{T} = \frac{d\alpha_i}{dt}\frac{\partial\Psi}{\partial\alpha_i}$. However, we follow the standard formalism in terms of *arc* length s.

Before we reproduce the Srret-Frenet's formulae in the context of Quantum Mechanics, we quickly discuss the settings of basic tenets of the Differential Geometry. We introduce *arc* length s, as a function of time, then velocity vector could be written as:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\hat{\sigma}; \tag{9}$$

where, $v = \frac{ds}{dt}$ is the absolute value of velocity and $\hat{\sigma}$ is the unit tangent vector directed along tangent line, thus, we can also denote it by $\hat{\mathbf{T}}$. Later, we replace vector $\mathbf{r}$ by unit (dimensionless) vector $\Psi \in \mathscr{H}$ for the present formulation and thus velocity should be deemed to be $\mathbf{v} = \frac{d\Psi}{dt} = \frac{d\Psi}{ds}\frac{ds}{dt}$. The unit vector $\hat{\mathbf{T}}$ is vital in the formulation of $Serret-Frenet$ formulae from first principles of differential geometry. One may define acceleration as:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\hat{\mathbf{T}})}{dt}. \tag{10}$$

Expanding this derivative we get,

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{T}} + v\frac{d\hat{\mathbf{T}}}{dt}. \tag{11}$$

We now transform the derivative $\frac{d\hat{\mathbf{T}}}{dt}$ as:

$$\frac{d\hat{\mathbf{T}}}{dt} = \frac{d\hat{\mathbf{T}}}{ds}\frac{ds}{dt} = \frac{v}{R}\hat{\mathbf{n}}. \tag{12}$$

Thus, we get the first $Serret - Frenet$ formula as:

$$\frac{d\hat{\mathbf{T}}}{ds} = \kappa\hat{\mathbf{n}}. \tag{13}$$

Where,

$$\kappa = \frac{1}{R}; \tag{14}$$

is curvature and R is the radius of curvature. Eq. (17) represents first of the three $Serret - Frenet$ formulae. Alternatively, the curvature κ could also be given as magnitude of the derivative of unit tangent vector with respect to *arc* parameter s. The other two $Serret - Frenet$ formulae are given as:

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau\hat{\mathbf{n}}; \tag{15}$$

and

$$\frac{d\hat{\mathbf{n}}}{ds} = -\tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{T}}. \tag{16}$$

Eq. (18) and (19) represent second and third $Serret - Frenet$ formulae respectively. Where, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ are normal and binormal, and κ and τ are curvature and torsion respectively. Also, this is to clarify that normal to a curve which is perpendicular to the osculating plane is called binormal. In the present paper curvature and torsion are described in the usual geometric sense. The physical interpretation is duly emphasized wherever it is warranted.

We now recast the $Serret - Frenet$ formulae in the context of quantum evolution in the following discussion with emphasis on its relevance in a dynamical scenario. We begin with the tangent space formalism which is nicely discussed in the context of dynamics using first principles of variational Calculus in various texts such as Pishkunov [18]. We now define a unit vector called binormal $\hat{\mathbf{b}}$ as:

$$\hat{\mathbf{T}} \times \hat{\mathbf{n}} = \hat{\mathbf{b}}; \tag{17}$$

obviously,

$$\hat{\mathbf{b}}.\hat{\mathbf{b}} = 1. \tag{18}$$

Thus, the derivative of binormal $\frac{d\hat{\mathbf{b}}}{ds}$ is found to be

$$\frac{d\hat{\mathbf{b}}}{ds} = \frac{d(\hat{\mathbf{T}} \times \hat{\mathbf{n}})}{ds} = \frac{d\hat{\mathbf{T}}}{ds} \times \hat{\mathbf{n}} + \hat{\mathbf{T}} \times \frac{d\hat{\mathbf{n}}}{ds}. \tag{19}$$

But,

$$\frac{d\hat{\mathbf{T}}}{ds} = \frac{1}{R}\hat{\mathbf{n}}; \tag{20}$$

therefore,

$$\frac{d\hat{\mathbf{T}}}{ds} \times \hat{\mathbf{n}} = \frac{1}{R}\hat{\mathbf{n}} \times \hat{\mathbf{n}} = 0; \tag{21}$$

and thus,

$$\frac{d\hat{\mathbf{b}}}{ds} = \hat{\mathbf{T}} \times \frac{d\hat{\mathbf{n}}}{ds}. \tag{22}$$

This relation is known as second $Serret - Frenet$ formula.
Let us note that $\hat{\mathbf{T}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ are all perpendicular to each other and also, $\frac{d\hat{\mathbf{T}}}{ds}$ is perpendicular to $\hat{\mathbf{b}}$. We denote length of the vector $\frac{d\hat{\mathbf{b}}}{ds}$ by τ as:

$$\mid \frac{d\hat{\mathbf{b}}}{ds} \mid = \tau; \tag{23}$$

here, τ is known as torsion and

$$\frac{d\hat{\mathbf{b}}}{ds} = \tau \times \hat{\mathbf{n}}. \tag{24}$$

Now, curvature in Differential Geometry in terms of vector $\Psi(t)$ can be given by the following expression:

$$\kappa^2 = \left[\frac{d^2\Psi}{dt^2}\frac{1}{(\frac{ds}{dt})^2} - \frac{d\Psi}{dt}\frac{\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^2}\right]^2 = \frac{[\frac{d\Psi}{dt} \times \frac{d^2\Psi}{dt^2}]^2}{[(\frac{d\Psi}{dt})^2]^3}. \tag{25}$$

It is pertinent to mention that this expression of curvature represents scalar curvature. In the structure of $Serret - Frenet$ formulae in Differential Geometry, curvature is a dimensionless quantity expressed in terms of *arc* length s and the length of the curve does not depend upon the ways and means of parametrization. For the sake of continuity of the discussion we discuss the remaining niceties in the Appendix of the paper.
Pishkunov [16] and various other texts have described curvature κ in terms of *arc* length s and even in terms of parameter t also. If parameter t is time, the consequent description is geometrodynamics and holds for dynamical systems. We now calculate curvature κ in quantum evolution of a state vector $\Psi(t)$ for an exponential family of curves such as:

$$\Psi(t) = \Psi(0)e^{-\frac{iH}{\hbar}t}. \tag{26}$$

Here, the function $\Psi(t)$ follows the evolution

$$\hat{H} \mid \Psi\rangle = E \mid \Psi\rangle. \tag{27}$$

We notice that

$$\left(\frac{d\Psi}{dt}\right)^2 = \left(\frac{d\Psi}{dt}\right)^* \left(\frac{d\Psi}{dt}\right) \Rightarrow \left(\frac{1}{\hbar}\right)^2 \langle\Psi \mid H^2 \mid \Psi\rangle = \left(\frac{1}{\hbar}\right)^2 \langle H^2\rangle; \tag{28}$$

and

$$\left(\frac{d^2\Psi}{dt^2}\right)^2 = \left(\frac{d^2\Psi}{dt^2}\right)^* \left(\frac{d^2\Psi}{dt^2}\right) \Rightarrow \left(\frac{1}{\hbar}\right)^4 \langle\Psi \mid H^4 \mid \Psi\rangle = \left(\frac{1}{\hbar}\right)^4 \langle H^4\rangle. \tag{29}$$

For the numerator in the eq. (28), we use the vector identity

$$\left(\frac{d\Psi}{dt} \times \frac{d^2\Psi}{dt^2}\right)^2 = \left(\frac{d\Psi}{dt}\right)^2 \left(\frac{d^2\Psi}{dt^2}\right)^2 - \left(\frac{d\Psi}{dt}\frac{d^2\Psi}{dt^2}\right)^2 \Rightarrow \left(\frac{1}{\hbar}\right)^6 \left(\langle H^2\rangle\langle H^4\rangle - \langle H^3\rangle^2\right). \tag{30}$$

Hence, the expression of curvature in the evolution of a vector Ψ in the eq. (28) turns out to be a dimensionless quantity

$$\kappa^2 = \frac{\langle H^2\rangle\langle H^4\rangle - \langle H^3\rangle^2}{\langle H^2\rangle^3} = \frac{\langle H^4\rangle}{\langle H^2\rangle^2} - \frac{\langle H^3\rangle^2}{\langle H^2\rangle^3}. \tag{31}$$

This is neverthless the same expression which is deduced for curvature of the exponential familty of curves in the quantum evolution by Brody and Houghston [1]. Thus, the present exercise further verifies and consolidates foundations of Geometric Quantum Mechanics.

Brody and Houghston [1] carried out a geometric formulation of parameter estimation based on the Geometry of Hilbert space. In the exercise by Brody and Houston [1] too, the curvature κ is in the literal sense curvature of a curve in the Hilbert space generated by a unitary evolution, and is calculated by the squared-length of the 'acceleration' vector, that is component of the second derivative of the state vector in time which is orthogonal to the zeroth and first derivatives of the vector.

Interestingly, the ratios $\frac{\langle H^4\rangle}{\langle H^2\rangle^2}$ and $\frac{\langle H^3\rangle^2}{\langle H^2\rangle^3}$ were known to staticians since long back [19-20] as *kurtosis* and *skewness* respectively in the estimator problem of Statistics. The first term in the expression of curvature κ in eq. (34) is identified as *kurtosis* that is measure of sharpness of the distribution, while the second term is found to be *skewness* that is measure of asymmetry [1, 19-20].

We emphasize here that the expectation values $\langle H^4\rangle$ and $\langle H^2\rangle^2$ ought to be of similar order and of considerable magnitude. However, the size of κ depends on the type of distribution [23]. In particular, even if it is the case that the two terms in κ are of similar order, since κ^2 is difference of two terms and not the sum, it can still be small [23].

We now compute torsion τ for quantum evolution of a vector Ψ, which in Differential Geometry appears in the following dimensionless expression:

$$\tau^2 = \left[\frac{\frac{d\Psi}{ds}\cdot\left(\frac{d^2\Psi}{ds^2} \times \frac{d^3\Psi}{ds^3}\right)}{\kappa^2}\right]^2 = \frac{\left[\frac{d\Psi}{dt}\cdot\left(\frac{d^2\Psi}{dt^2} \times \frac{d^3\Psi}{dt^3}\right)\right]^2}{\left[\left(\frac{d\Psi}{dt} \times \frac{d^2\Psi}{dt^2}\right)^2\right]^2}. \tag{32}$$

It is worth noticing that the first derivative of the vector is along the tangent and the vector perpendicular to the plane of the second and the third derivatives also lies along the direction of the tangent.

As expected, torsion depends on the curvature, and therefore curvature and torsion both are strongly correlated. The term in the denominator of eq. (34) for the quantum states of the type $\Psi(t) = \Psi(0)e^{-\frac{iH}{\hbar}t}$, is already deduced as: $\left(\frac{d\Psi}{dt} \times \frac{d^2\Psi}{dt^2}\right)^2 = \left(\frac{d\Psi}{dt}\right)^2 \left(\frac{d^2\Psi}{dt^2}\right)^2 - \left(\frac{d\Psi}{dt}\frac{d^2\Psi}{dt^2}\right)^2 \Rightarrow \left(\frac{1}{\hbar}\right)^6 \left(\langle H^2\rangle\langle H^4\rangle - \langle H^3\rangle^2\right)$.

We now evaluate the term in the numerator of eq. (34) as

$$\left(\frac{d\Psi}{dt}\right)^2 \left[\frac{d^2\Psi}{dt^2} \times \frac{d^3\Psi}{dt^3}\right]^2 = \left(\frac{d\Psi}{dt}\right)^2 \left[\left(\frac{d^2\Psi}{dt^2}\right)^2 \left(\frac{d^3\Psi}{dt^3}\right)^2 - \left(\frac{d^2\Psi}{dt^2}\frac{d^3\Psi}{dt^3}\right)^2\right] \Rightarrow \left(\frac{1}{\hbar}\right)^{12} \left[\langle H^2\rangle \left(\langle H^4\rangle\right)\langle H^6\rangle - \langle H^5\rangle^2)\right]. \tag{33}$$

Thus, torsion in a dimensionless expression could be expressed as:

$$\tau^2 = \frac{\left[\langle H^2\rangle \left(\langle H^4\rangle\langle H^6\rangle - \langle H^5\rangle^2\right)\right]}{\left[\langle H^2\rangle\langle H^4\rangle - \langle H^3\rangle^2\right]^2}. \tag{34}$$

In fact, expectation values of the odd order for symmetric states vanish as:

$$\langle H^5\rangle = \langle H^3\rangle = 0. \tag{35}$$

Consequently, the torsion during evolution of symmetric states turns out to be:

$$\tau^2 = \frac{\langle H^6\rangle}{\langle H^2\rangle\langle H^4\rangle}. \tag{36}$$

In the present context, by symmetric states we mean the states that follow Schrödinger's evolution. In statistical exposition, the symmetric functions are those functions that have symmetric distribution about the central value. The expectation value $\langle H\rangle$ being the central expectation value does not vanish, and hence,

$$\langle H\rangle \neq 0. \tag{37}$$

It is important to note that the expression of curvature in the equation (34) and the expression of torsion in the eq. (39) are relevant for the quantum states that follow Schrödinger's evolution. Whereas, the expression of curvature in the eq. (28) as well as the expression of torsion in the eq. (35) are both valid for all state functions in general.

IV. CURVATURE AND TORSION IN THE NEIGHBOURHOOD TEST

We have precedence, where neighborhood test was applied to explore the quantum evolution and it resulted into valuable information in the form of distance and metricity in quantum evolution. We find it to be unlimited treasure of information particularly when we explore it up to fourth and sixth order. We consider the quantum states of the type $\Psi(t) = \Psi(0)e^{-\frac{iH}{\hbar}t}$ that follow the evolution $\hat{H} \mid \Psi\rangle = E \mid \Psi\rangle$, and examine the Taylor's expansion of the function $\Psi(t+dt)\rangle$ for the infinitesimal time evolution as:

$$\Psi(t+dt)\rangle = \Psi(t)\rangle + \frac{d}{dt}\Psi(t)\rangle dt + \frac{d^2}{dt^2}\Psi(t)\rangle\frac{dt^2}{2!} + \frac{d^3}{dt^3}\Psi(t)\rangle\frac{dt^3}{3!} + \frac{d^4}{dt^4}\Psi(t)\rangle\frac{dt^4}{4!} + \frac{d^5}{dt^5}\Psi(t)\rangle\frac{dt^5}{5!} + \frac{d^6}{dt^6}\Psi(t)\rangle\frac{dt^6}{6!} + \ldots\ldots\ldots\ldots \tag{38}$$

Taking inner product of the eq. (42) with $\langle\Psi(t) \mid$, we get

$$\begin{aligned}\langle\Psi(t) \mid \Psi(t+dt)\rangle = \langle\Psi \mid \Psi\rangle + \left(\frac{-i}{\hbar}\right)\langle\Psi \mid H \mid \Psi\rangle dt + \left(\frac{-i}{\hbar}\right)^2\langle\Psi \mid H^2 \mid \Psi\rangle\frac{dt^2}{2!} + \left(\frac{-i}{\hbar}\right)^3\langle\Psi \mid H^3 \mid \Psi\rangle\frac{dt^3}{3!}\\ + \left(\frac{-i}{\hbar}\right)^4\langle\Psi \mid H^4 \mid \Psi\rangle\frac{dt^4}{4!} + \left(\frac{-i}{\hbar}\right)^5\langle\Psi \mid H^5 \mid \Psi\rangle\frac{dt^5}{5!} + \left(\frac{-i}{\hbar}\right)^6\langle\Psi \mid h^6 \mid \Psi\rangle\frac{dt^6}{6!} + \ldots.\end{aligned} \tag{39}$$

If we take magnitude of this expression, square it and subtract it from unity we get

$$\left(1- \mid \langle\Psi(t) \mid \Psi(t+dt)\rangle \mid^2\right) = \frac{1}{\hbar^2}\left[\langle H^2\rangle - \langle H\rangle^2\right]dt^2 + \frac{1}{\hbar^4}\left[\frac{\langle H^4\rangle}{12} - \frac{\langle H^2\rangle^2}{4}\right]dt^4 + \frac{1}{\hbar^6}\left[\frac{2\langle H^6\rangle}{6!} - \frac{\langle H^2\rangle\langle H^4\rangle}{4!}\right]dt^6 + \ldots\ldots.. \tag{40}$$

It should be noticed that for functions with symmetric distribution, the odd order expectation values vanish as: $\langle H^5\rangle = \langle H^3\rangle = 0$. In the present context, by symmetric states we mean the states that follow Schrödinger's evolution. And the expectation value $\langle H\rangle$ being the central expectation value does not vanish as: $\langle H\rangle \neq 0$. The second order term in the expression in eq. (44) is identified [2-7] as invariant: $ds^2 = \Delta E^2 dt^2$. When generalized on the space of quantum states this is called metric of quantum evolution.
In the eq.(44) if we pull out the second terms in the parentheses in fourth and sixth order terms, we find

$$\left(1- \mid \langle\Psi(t) \mid \Psi(t+dt)\rangle \mid^2\right) = \frac{1}{\hbar^2}\left[\langle H^2\rangle - \langle H\rangle^2\right]dt^2 + \frac{\langle H^2\rangle^2}{12\hbar^4}\left[\frac{\langle H^4\rangle}{\langle H^2\rangle^2} - 3\right]dt^4 + \frac{\langle H^2\rangle\langle H^4\rangle}{4!\hbar^6}\left[\frac{\langle H^6\rangle}{\langle H^2\rangle\langle H^4\rangle} - 15\right]dt^6 + \ldots\ldots.. \tag{41}$$

As discussed earlier, the dimensionless coefficient in the second order term appearing on the right side of eq. (44) $ds^2 = \frac{1}{\hbar^2}\left[\langle H^2\rangle - \langle H\rangle^2\right]dt^2 = \Delta E^2 dt^2$
is called metric of the quantum states [2-7]. The dimensionless coefficient in the fourth order term can be easily identified with curvature as:

$$\kappa^2 = \left(\frac{\langle H^4\rangle}{\langle H^2\rangle^2} - 3\right); \tag{42}$$

Wherein, skewness for symmetric functions vanishes as $\frac{\langle H^3\rangle^2}{\langle H^2\rangle^3} = 0$.
The numerical factor 3 also conveys a message that we ellaborate in the following paragraphs. Whereas the dimensionless coefficient in the sixth order term could be identified as torsion:

$$\mathfrak{T} = \left(\frac{\langle H^6\rangle}{\langle H^2\rangle\langle H^4\rangle} - 15\right) = \tau^2 - 15. \tag{43}$$

Here too, the numerical factor 15 carries a message that is well understood in Statistics. Interestingly, numerical factors 3 and 15 in the expressions of curvature and torsion respectively are already well understood in the studies of fourth and sixth order moments in Statistics. However, we know from our experience of direct formulation of torsion by differential geometric means in this paper and from the reference [1] that expression in eq. (46) unambigously represents torsion. The definition of *kurtosis* [19, 20] in its entirety is given in two ways:

$$\beta_2 = \frac{\langle H^4 \rangle}{\langle H^2 \rangle^2}; \tag{44}$$

and

$$\gamma_2 = \beta_2 - 3 = \frac{\langle H^4 \rangle}{\langle H^2 \rangle^2} - 3. \tag{45}$$

This simply implies three possibilities *viz.*,

$$\gamma_2 \begin{cases} > 0 & \text{if } \frac{\langle H^4 \rangle}{\langle H^2 \rangle^2} > 3, \ \beta_2 \text{ is said to be platy-kurtic,} \\ = 0 & \text{if } \frac{\langle H^4 \rangle}{\langle H^2 \rangle^2} = 3, \ \beta_2 \text{ is said to be meso-kurtic.} \\ < 0 & \text{if } \frac{\langle H^4 \rangle}{\langle H^2 \rangle^3} < 3, \ \beta_2 \text{ is said to be lepto-kurtic} \end{cases} \tag{46}$$

The expression of curvature in the present case implies that

$$\kappa^2 = \left(\frac{\langle H^4 \rangle}{\langle H^2 \rangle^2} - 3 \right) \neq 0, \tag{47}$$

as the *kurtic* value $\beta_2 + 3$ is not zero.
In a similar note, we realize the importance of numerical factor 15 in the expression of torsion in eq. (46). For the sixth order moments there can be 15 terms of permutation [21, 22], and there can be three possibilities with the expression of torsion:

$$\mathfrak{T} \begin{cases} > 0 & \text{if } \frac{\langle H^6 \rangle}{\langle H^2 \rangle \langle H^4 \rangle} > 15, \\ = 0 & \text{if } \frac{\langle H^6 \rangle}{\langle H^2 \rangle \langle H^4 \rangle} = 15. \\ < 0 & \text{if } \frac{\langle H^6 \rangle}{\langle H^2 \rangle \langle H^4 \rangle} < 15 \end{cases} \tag{48}$$

V. SUMMARY AND DISCUSSION

In the present exercise curvature κ and torsion τ in the quantum evolution are discussed and the Serret-Frenet formulae are recast in terms of *arc* length s using the first principles of Differential Geometry. Curvature κ in quantum evolution has been formulated [1] earlier too, and thus evaluation of curvature in quantum evolution is reassuring in more than one ways. Whereas, the present exercise is the first ever attempt to furmulate torsion in quantum evolution. For illustration quantum state function of simple exponential form has been considered as usual, and there is scope for examination of quantum states of various types in the Future.
This physical interpretation of the curvature and torsion is in tune with the mathematical spirit. The curvature indicates a deviation of the state vector of the evolution from the geodesic line whereas torsion indicates deviation of state vector from the plane of evolution at a given time. Thus, torsion is measure of deviation of a curve from the plane of the curve itself. If a curve is plane curve then the osculating plane does not change its direction, and the torsion is zero. The torsion expressed in terms of curvature indicates that as usual curvature and torsion are strongly correlated.
In the present exercise on neighbourhood test, we do not limit the investigation to second order terms and extend our exploration to the fourth and sixth order terms and as a result of this we harvest valuable information.

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[1] D. C. Brody and L. P. Houston, Phys. Rev. Lett. **77**, No 14, 2851 (1996); Proc. R. Soc. Lond. **A 454**, 2445 (1998).
[2] V. M. Tkachuk,arXiv:1006.4447[quant-ph].
[3] Aalok, Int. J. Theor. Phys. **48**, 522 (2009); Int. J. Theor. Phys. **46**, 3216 (2007).
[4] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
[5] J. Anandan, Phys. Lett. **A 147**, 3 (1990); Found. of Phys. 21, 1265 (1991).
[6] J.P. Provost and G. Vallee, Comm. Math. Phys. **76**, 289 (1980).
[7] V. Bargmann, J. Math. Phys. **5**, 862 (1962).
[8] M. V. Berry, Proc. R. Soc. Lond. **5**, 862 (1983).
[9] A. K. Pati, Phys. Lett. **A 159**, 105 (1991); J. Phys. **A 25**: Math. and Gen. L1001 (1992); J. Phys. **42** 455 (1994); A. K. Pati and A. Joshi, Phys. Rev. **A 47**, 98 (1993).
[10] T. W. Kibble, Comm. Math. Phys. **65**, 189 (1979).
[11] Bei Jia and Xi-guo Li: math-ph/0701011 (2007).
[12] D. C. Brody and L. P. Houston, J. Geom. Phys. **38**, 19 (2001).
[13] R. F. Timothy and L. P. Houston, J. Math. Phys. **40**, 2539 (1999).
[14] V. Jejjala and D. Minic: hep-th/0605105 (2006); V. Jejjala, D. Minic and C. H. Tze: gr-qc/0406037 (2004).
[15] D. Minic and C. H. Tze, Phys. Lett. **A 536**, 305 (2002); hep-th/0202173 (2002); hep-th/0401028 (2004).
[16] D. Minic and C. H. Tze, Phys. Rev. **D 68**, 061501 (2003); hep-th/0305193 (2003).
[17] D. Minic and C. H. Tze, Phys. Lett. **B 581**, 111 (2004); hep-th/0309239 (2003).
[18] N. Pishkunov, Differential and Integral Calculus (Vol. I) Mir Publishers, Moscow, pp. 325-334 (1977).
[19] A. Bhattacharya, Sankhya **8**, 1 (1946).
[20] Encyclopedia of Statistical Sciences **3** (2nd Ed.), 1630 (2006) Wiley Interscience. http://www.mrw.interscience.wiley.com/ess
[21] A. M. Goon., M. K. Gupta and B. Dasgupta: Fundamentals of Statistics (Vol. I), World Press (1998); Fundamentals of Statistics (Vol. I), World Press (2001).
[22] L. Isserlis, Biometrika, **12**, 134 (1918).
[23] D. C. Brody (Brought out these niceties in private communication).