Some physics of the kinetic-conformal Horava theory

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1. Definition of the theory and its canonical formulation

With the aim of obtaining simultaneously perturbative renormalizability and unitarity, Hořava[1] defines a gravitational theory with higher orders only in spatial derivatives, breaking the symmetry of general diffeomorphisms over the spacetime. Indeed, the very concept of spacetime is substituted by the one of foliation of space-like hypersurfaces along an absolute line of time. The gauge symmetry is given by the diffeomorphisms that preserve the foliation (FDiff). The theory is defined in terms of the Arnowitt-Deser-Misner variables N, N_i and g_{ij}. They are understood as tensors over the hypersurfaces that evolve in time. The Lagrangian has a kinetic terms that is of second order in time derivatives,

$$\mathcal{L}_K = \sqrt{g} N G^{ijkl} K_{ij} K_{kl} \,, \quad G^{ijkl} \equiv \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl} \,, \tag{1}$$

and K_{ij} is the extrinsic curvature of the hypersurfaces. This kinetic term is Fdiff covariant for any value of the dimensionless coupling constant λ. Our concern in this paper is a particular formulation of the Hořava theory given by a critical value of λ, which we call the kinetic-conformal theory. For spatial hypersurfaces of dimension 3, the critical value for λ we refer to is $\lambda = 1/3$. This value defines a dynamically different formulation of the Horava theory in the sense that the structure of constraints is discontinuous to the generic formulation with $\lambda \neq 1/3$; it cannot be obtained by continuously varying λ. At $\lambda = 1/3$ the hypermatrix G^{ijkl} becomes degenerated and this leads to the raising of the primary constraint $\pi \equiv g_{ij}\pi^{ij} = 0$, where π^{ij} is the canonically conjugate of g_{ij}. There is also an additional secondary constraint that emerges when the time preservation of π is imposed. Then the kinetic-conformal theory propagates less physical degrees of freedom than the generic formulation. It propagates two physical modes, the same number of General Relativity. The so-called extra mode of the generic formulation of the Hořava theory is absent in the kinetic-conformal formulation. We consider this an interesting feature that deserves to be explored. Furthermore, at $\lambda = 1/3$ the kinetic term (1) gets an *anisotropic* conformal symmetry defined by the anisotropic

Weyl transformations[1]

$$\tilde{g}_{ij} = \Omega^2 g_{ij}\,, \qquad \tilde{N} = \Omega^3 N\,, \qquad \tilde{N}_i = \Omega^2 N_i\,, \tag{2}$$

where $\Omega = \Omega(t,\vec{x})$. This motivates the name kinetic-conformal. The full theory is not conformally invariant since the potential is not, unles it is defined with specific conformal terms.

In Ref.[11] several versions of the Hořava theory were associated to the algebra of the Newton-Cartan geometry. The kinetic-conformal formulation can be found within this program by fixing the values of some coupling constants on the side of the Newton-Cartan dynamics. The extrinsic curvature arises via the covariant derivatives of the inverse vielbein denoted by $\hat{v}^\mu$ in Ref.[11]. Then the kinetic term (1) emerges from the terms that are quadratic in derivatives of $\hat{v}^\mu$. There are two such terms in the Newton-Cartan action. Their coupling constants, denoted by c_3 and c_4 in Ref.[11], determine the constant λ. Therefore, the value $\lambda = 1/3$ is achieved by adjusting c_3 and c_4.

The action of the nonprojectable Hořava theory is[1,2]

$$S = \int dt d^3x \sqrt{g} N \left(G^{ijkl} K_{ij} K_{kl} - \mathcal{V} \right), \tag{3}$$

where to define the kinetic-conformal theory we consider that the value $\lambda = 1/3$ has been fixed. The potential $\mathcal{V}$ should include all the inequivalent terms that are FDiff covariant and up to sixth order in spatial derivatives ($z = 3$ terms), as required for the power-counting renormalizability[1]. These terms are[7]

$$-\mathcal{V}^{(z=1)} = \beta R + \alpha a_i a^i\,, \tag{4}$$

$$-\mathcal{V}^{(z=2)} = \alpha_1 R \nabla_i a^i + \alpha_2 \nabla_i a_j \nabla^i a^j + \beta_1 R_{ij} R^{ij} + \beta_2 R^2\,, \tag{5}$$

$$-\mathcal{V}^{(z=3)} = \alpha_3 \nabla^2 R \nabla_i a^i + \alpha_4 \nabla^2 a_i \nabla^2 a^i + \beta_3 \nabla_i R_{jk} \nabla^i R^{jk} + \beta_4 \nabla_i R \nabla^i R\,, \tag{6}$$

where $a_i = \partial_i \ln N$.

In order to determine the dynamical consistency of the theory we have performed its Hamiltonian formulation[3]. The phase space is spanned by the conjugate pairs (g_{ij}, π^{ij}) and (N, P_N). The Hamiltonian, with the primary constraints added, is given by

$$H = \int d^3x \left(\frac{N}{\sqrt{g}} \pi^{ij} \pi_{ij} + \sqrt{g} N \mathcal{V} + N_i \mathcal{H}^i + \mu \pi + \sigma P_N \right), \tag{7}$$

and the full set of constraints is

$$\mathcal{H}^j \equiv -2\nabla_i \pi^{ij} + P_N \partial^j N = 0\,. \tag{8}$$

$$P_N = 0\,, \quad \pi = 0\,, \tag{9}$$

$$\frac{1}{\sqrt{g}} \mathcal{H} \equiv \frac{1}{g} \pi^{ij} \pi_{ij} - \beta R + 2\alpha \frac{\nabla^2 N}{N} - \alpha a_i a^i = 0\,, \tag{10}$$

$$\frac{1}{\sqrt{g}} \mathcal{C} \equiv \frac{3}{2g} \pi^{ij} \pi_{ij} + \frac{\beta}{2} R - 2\beta \frac{\nabla^2 N}{N} + \frac{\alpha}{2} a_i a^i = 0\,. \tag{11}$$

$\mathcal{H}^i$ is a first-class constraint whereas the four constraints P_N, π, $\mathcal{H}$ and $\mathcal{C}$ are of second class. In the Hamiltonian formulation the shift vector N_i plays the role of Lagrange multiplier (as in GR), as well as μ and σ. The time preservation of the second-class constraints leads to equations for μ and σ. When the complete potential potential $\mathcal{V}$ is considered, it can be shown[3] that these are elliptic (sixth-order) equations, hence they can be consistently solved with appropiated boundary conditions. This ends the procedure of Dirac, the structure of constraints is closed. The constraints $\mathcal{H}$ and $\mathcal{C}$ can also be casted as elliptic equations if the appropiated field variables are chosen to solve them, see[4]. This analysis shows that the Hamiltonian formulation of the theory is consistent. Considering the second-class nature of P_N, π, $\mathcal{H}$ and $\mathcal{C}$, it results that the theory propagates two physical modes, coinciding with the number of modes in GR.

The presence of the $\pi = 0$ constraint is intriguing, since it generates the Weyl scalings on g_{ij} and π^{ij}, but this theory is not conformally invariant. This is in agreement with the fact that π is of second-class, hence it is not the generator of gauge symmetries. Contrasting with a purely anisotropic conformal Horava theory, which is given by a conformal potential, we have that $\pi = 0$ can be combined with the $P_N = 0$ constraint to form the full generator of the anisotropic conformal transformations (2), which is

$$\varpi = \pi + \frac{3}{2} N P_N \,. \tag{12}$$

In the conformal theory this is a first-class constraint. It is preserved without further conditions, hence no furhter constraints are generated. Therefore, in the exact conformal case there is a symmetry more than in the kinetic-conformal formulation, the anisotropic Weyl scalings, but a constraint less (the $\mathcal{C}$), hence the number of physical degrees of freedom is the same in both cases, and it is the same of GR. It is interesting to further explore the similarities between the kinetic-conformal theory and the exact conformal formulation. Some of us are currently considering this study[6].

2. Quantization: propagators and the superficial degree of divergence

A perturbative analysis considering all the terms (4) - (6) allows to check explicitly the consistency of the Hamiltonian formulation of the theory and to obtain the propagators of the physical modes. We impose the transverse gauge $\partial_i h_{ij} = 0$, where h_{ij} represents the perturbative metric around the Minkowski background. The momentum constraint eliminates the longitudinal sector of the canonical momentum at first order in perturbations. Constraints $\mathcal{H}$ and $\mathcal{C}$ are consistently solved for h_{kk} and the pertubative version of the lapse function, n, fixing these variables to zero at first order in perturbations (with asymptoticallt flat boundary conditions). Constrain $\pi = 0$ elminates the trace of the canonical momentum. There remains the transverse-traceless sector h_{ij}^{TT} as the independent propagating physical modes (in the transverse gauge). The corresponding propagators are[4]

$$\langle h_{ij}^{TT} h_{kl}^{TT} \rangle = \frac{P_{ijkl}^{TT}}{\omega^2 - \beta \vec{k}^{\,2} + \beta_1 \vec{k}^{\,4} + \beta_3 \vec{k}^{\,6}} \,, \tag{13}$$

where P_{ijkl}^{TT} is the transverse-traceless projector.

Another important issue of the perturbative quantization is the distribution of Fourier momentum in Feynmann diagrams. We recall that this is a theory with second-class constraints, hence standard techniques of gauge field theories which only have first-class constraints do not apply. One plausible scheme of quantizaton is to solve the second-class constraints perturbatively. It can be shown that the perturbative field variables used to solve the constraints end with a balance of zero Fourier momentum[4], hence when the solutions are substituted in the Lagrangian they do not alter the order in momentum of the interacting terms, the weight in Fourier momentum of the vertices is the same of the off-shell theory. These results allow to evaluate the superficial degree of divergence of Feynman diagrams, since the badly divergent diagrams are those with vertices of highest order, which is the sixth order according to the power counting of the Lagrangian. The internal lines scale also with sixth order according to the propagator (13). With these considerations we may show[4] that the superficial degree of divergence of the badly divergent diagrams is given by the order 6. This implies that counter-terms of 6th order in spatial derivatives must be added to the bare Lagrangian, but 6 is precisely the order of the bare Lagrangian designed for power-counting renormalizability. Thus, the theory passes the criterium given by the superficial degree of divergence needed for the renormalization of the theory.

3. Gravitational waves and observational bounds

The coincidence in the number physical degrees of freedom between the kinetic-conformal Hořava theory and GR raises interest on the behavior of the gravitational waves in this theory. This was analyzed in[5] using the equivalence of the large-distance effective action with the Einstein-aether theory[8]. The Einstein-aether theory implements the breaking of the Lorentz invariance keeping the gauge symmetry of general diffeomorphisms over spacetime by using a dynamic unit timelike vector (the aether). The analysis of gravitational waves done in[5] was achieved in a gauge-invariant way, involving the aether field in the construction of the gauge invariants. The first main result is that the transverse-traceless sector is propagated with a wave equation, $\sqrt{\beta}$ being the speed of the gravitational waves.

For the case of an isolated source, the dominant mode of its gravitational radiation in the far zone can be deduced by applying the same techniques of GR. The considerations on the source are the standard ones for a weak source: small mass, slow velocity and negligible self-gravity. If I_{ij} is the quadrupole moment given by the 00 component of the energy-momentum tensor, then the leading contribution for the generation of gravitational waves has the same structure of Einstein's quadrupole formula of GR,

$$h_{ij}^{TT} = \frac{\kappa_{\rm H}}{4\pi\beta r} P_{ijkl}^{TT} \frac{d^2 I_{kl}(t - r/\sqrt{\beta})}{dt^2} , \tag{14}$$

where $\kappa_{\rm H}$ is the coupling constant arising in front of the Hořava action (which we set equal to one in Eq. (3), since that is a vacuum action). To get an exact matching with the quadrupole formula of GR, we must set the coupling constants $\kappa_{\rm H}$ and β equal to their GR values, $\kappa_{\rm H} = 8\pi G_{\rm N}$ and $\beta = 1$.

We may apply the analysis of the parameterized-post-Newtonian (PPN) expansion for solar-system tests to the kinetic-conformal theory. It turns out that the theory reproduces the same values of the PPN parameters of GR, except for the parameters $\alpha_1^{\rm PPN}$ and $\alpha_2^{\rm PPN}$, whose deviations from the zero value indicate Lorentz-symmetry violation. For the kinetic-conformal Hořava theory these two constants are given by [5]

$$\alpha_2^{\rm PPN} = \frac{1}{8}\alpha_1^{\rm PPN} = \beta - 1 - \frac{\alpha}{2} . \tag{15}$$

The current observational bounds [10] on these parameters are $|\alpha_1^{\rm PPN}| < 10^{-4}$ and $|\alpha_2^{\rm PPN}| < 10^{-9}$. Relation (15) demands that the strong bound, which is the one on $\alpha_2^{\rm PPN}$, must be satisfied by both parameters. This condition is met if

$$\alpha = 2(\beta - 1) + \delta , \tag{16}$$

where δ represents the narrow observational window for the $\alpha_2^{\rm PPN}$ parameter, i. e. $|\delta| < 10^{-9}$.

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