Time slicings of black holes

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Introduction

Many descriptions and interpretations of Schwarzschild-Droste spacetime assume the *static* foliation (given by *t=const* in Schwarzschild-Droste coordinates), but do not explicitly say so. We investigate the spacetime splitting due to "moving" observers, a contrast which emphasises the *relativity* of black holes. This is important conceptually and pedagogically, and avoids "neo-Newtonian" (Eisenstaedt) misconceptions. It has also proven useful in research: Gullstrand-Painleve coordinates have led to an interpretation of Hawking radiation as quantum tunnelling (Parikh & Wilczek).

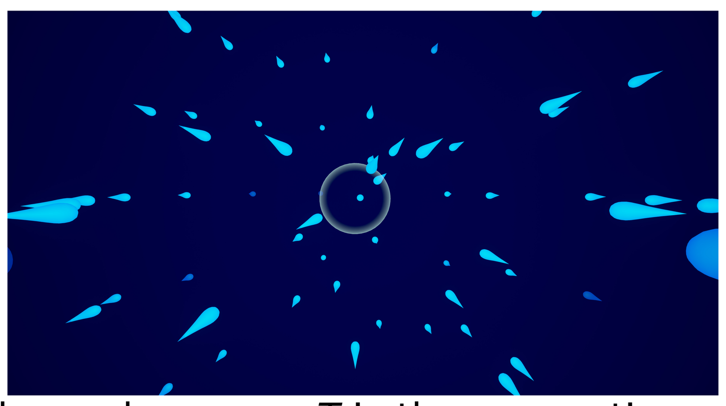
Observer families and coordinates

Suppose spacetime is filled by a family of test particles, all freefalling radially inward with the same "energy per mass" *e*:

$$e \equiv -\xi \cdot \mathbf{u}$$

u is the 4-velocity field, ξ is the Killing vector field which is timelike for all *r*>2*M*, and so our parameter *e* is the corresponding invariant along a geodesic. We can classify the observers into "rain" which fall from rest at infinity, "hail" which start from infinity with initial inward velocity, and "drips" which fall from rest at finite distance (Taylor & Wheeler). There exists another type I dub "snow", which falls more 'slowly' than the others inside the horizon.

Metaphor	Energy per mass	Allowed region
“hail”	$e > 1$	all
“rain”	$e = 1$	all
“drip”	$0 < e < 1$	$r \leq \frac{2M}{1-e^2}$
“mist”	$e \leq 0$	$r < 2M$



The following coordinate systems are well-suited to these observers; *T* is the proper time:

Generalised Gullstrand-Painleve coordinates:

$$ds^2 = -\frac{1}{e^2}\left(1-\frac{2M}{r}\right)dT^2 + \frac{2}{e^2}\sqrt{e^2-1+\frac{2M}{r}}\,dT\,dr + \frac{1}{e^2}dr^2 + r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

Generalised Lemaitre coordinates: $(-\infty < e < \infty,\ e \neq 0)$

$$ds^2 = -dT^2 + \frac{1}{e^2}\left(e^2-1+\frac{2M}{r}\right)d\rho^2 + r^2(d\theta^2+\sin^2\theta\,d\phi^2)$$

Use Schwarzschild coordinates for *e*=0 observers, since these are comoving (*t*=const).

Radial distance

One might assume the measurement of spatial distance in relativity is perfectly understood, but is it? The textbook radial "proper distance" is:

$$dL = \left(1-\frac{2M}{r}\right)^{-1/2}dr$$

However this is only the measurement of *stationary* observers (which have fixed r, including *static* observers). But for observers with arbitrary radial motion, this generalises to:

$$dL = \frac{1}{|e|}dr$$

(Gautreau & Hoffmann showed this for 0<*e*<1.) This is consistent with the previous result since static observers have:

$$e = \sqrt{1-\frac{2M}{r}}$$

The distance relates the *r*-coordinate to the observer's frame, so the slicing is not by Schwarzschild time but adapted to the observer. Importantly, the "radial" 4-direction depends on the observer; the mix of "space" and "time" is relative. The following approaches for *local* distance are identical:

- well-suited coordinates: if the time coordinate is proper time, set $dT = d\theta = d\phi = 0$
- spatial projector: $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$
- orthonormal tetrad frame
- radar metric $\gamma_{ij} \equiv g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}$

Caution: contracting the spatial metric tensor (projector or radar) with the "vector" (0,1,0,0) in Schwarzschild-Droste coordinates yields the following, which is the static observer's measurement of the moving observer's length:

$$dL = \left|e\left(1-\frac{2M}{r}\right)\right|dr$$

3-Volume

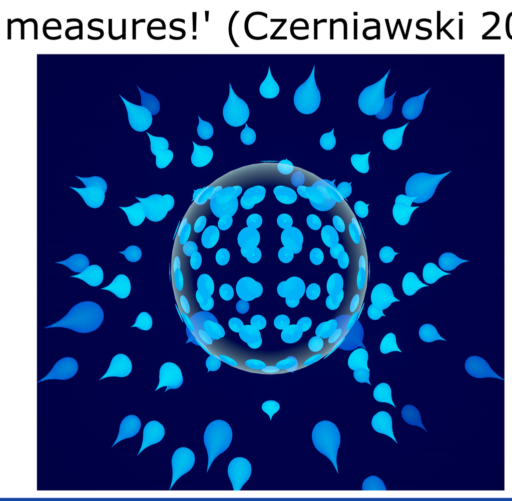
Our observers preserve spherical symmetry, so the spatial volume inside the event horizon is 4π times the radial distance: $\frac{1}{|e|}\frac{4\pi(2M)^3}{3}$

(Finch showed this for *e*>0.) More formally, this follows from the volume element:

$$\sqrt{|g_3|}d^3x = \frac{1}{|e|}\sqrt{e^2-1+\frac{2M}{r}}\,r^2\sin\theta\,d\rho\wedge d\theta\wedge d\phi$$

Time for a distant observer

The Schwarzschild-Droste *t*-coordinate is often called the "time at infinity", meaning the proper time of a static observer at spatial infinity. Historically this led to misconceptions including infinite time to fall into the interior, and time running backwards there. But at $r < \infty$ (indeed for observers at different spatial infinity) it depends on the simultaneity convention. For our slicing and $e \geq 1$, the time at infinity is 1/*e* *dT*. So to the question `How long is a raindrop's flight, for a distant observer?' we answer, in the *e*=1 case, `The same time as the raindrop itself measures!' (Czerniawski 2004)

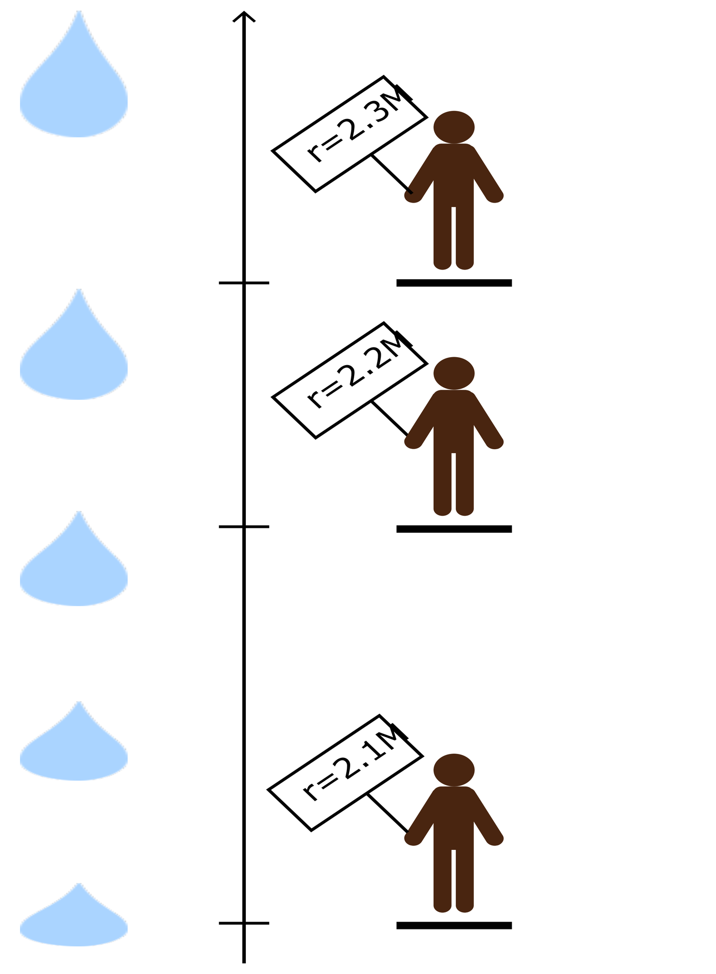


Length-contraction (and expansion)

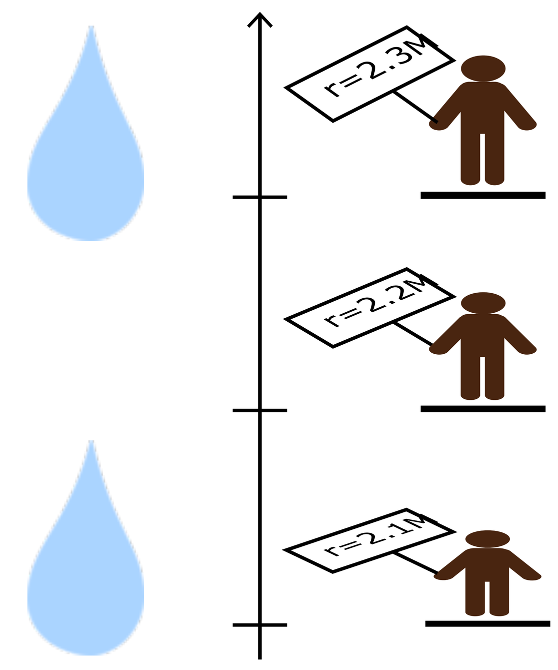
A moving and static observer have relative 3-velocity given by the Lorentz factor:

$$\gamma = e\left(1-\frac{2M}{r}\right)^{-1/2}$$

The radial lengths obtained previously are factors of γ and $\frac{1}{\gamma}$ relative to the static distance:



Static perspective: both the fallers and the distance between them are increasingly length-contracted as *r* decreases.



Moving perspective: both the static observers and the distance between them are increasingly length-contracted.

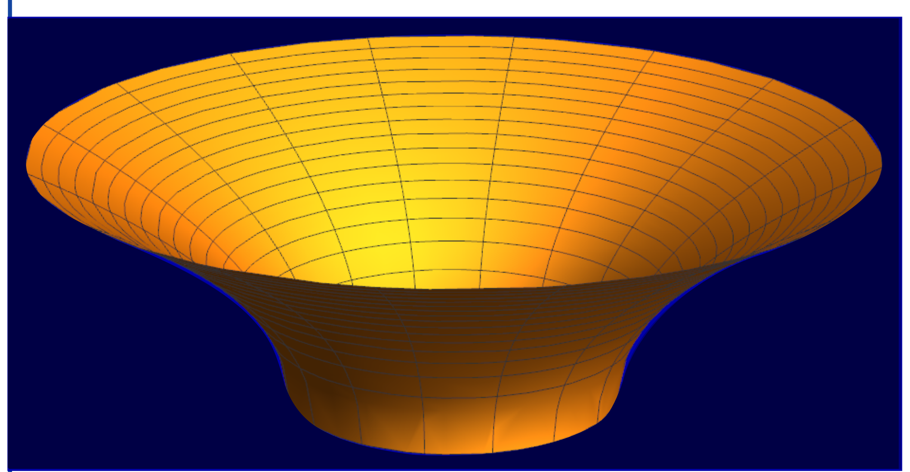
Curvature of 3-space

The curvature of space is often depicted by a 2D surface embedded in Euclidean $\mathbb{R}^3$ with the same curvature. Take a constant "time" slice $x^0 = const$ and equatorial slice $\theta = \frac{\pi}{2}$. In cylindrical coordinates (r,θ,z) the embedded surface *z=z(r)* has metric

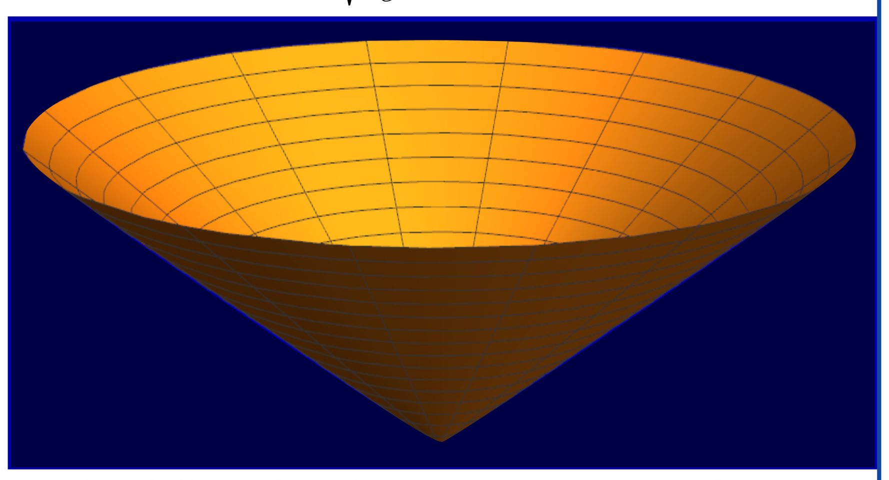
$$ds^2 = dx^2 + dy^2 + dz^2 = \left(1+\left(\frac{dz}{dr}\right)^2\right)dr^2 + r^2 d\phi^2$$

Comparing, we obtain:

$$z = \pm 2\sqrt{2M(r-2M)} \qquad\qquad z = \pm\sqrt{\frac{1}{e^2}-1}\,r \propto r$$



To static observers, space is a funnel, "Flamm's paraboloid".



To the falling family, space is a cone for |*e*|<1. For |*e*|=1 it is a flat plane (c.f. Lemaitre 1932), and for |*e*|>1 it cannot be depicted by this method.

Coordinate vectors and hypersurfaces

The description "time and space swap roles inside the horizon" has inspired eloquent prose. For a given coordinate $x^{(\alpha)}$ the coordinate surface has normal vector $n^\mu = g^{(\alpha)\mu}$ which has norm-squared $\mathbf{n}\cdot\mathbf{n} = g^{(\alpha)(\alpha)}$. This determines the timelike/null/spacelike nature of the coordinate. So for Schwarzschild-Droste coordinates, indeed *t* and *r* swap roles. However for our new coordinates, *T* remains timelike everywhere.

But the coordinate *vector* $\partial_{(\alpha)}$ has norm-squared $\partial_{(\alpha)}\cdot\partial_{(\alpha)} = g_{(\alpha)(\alpha)}$ which may have a different nature if the metric is not diagonal. For instance, the *r*-coordinate vector is:

Coordinates	Inner product $\partial_r \cdot \partial_r = g_{rr}$	Interpretation
Schwarzschild	$\left(1-\frac{2M}{r}\right)^{-1}$	spacelike for r > 2M timelike for r < 2M
generalised Gullstrand-Painlevé	$\frac{1}{e^2}$	spacelike
generalised Lemaître	$\frac{1}{e^2}(e^2-1+\frac{2M}{r})$	spacelike
Eddington-Finkelstein (null version)	0	null
Eddington-Finkelstein (timelike version)	$1+\frac{2M}{r}$	spacelike

Coordinate "vectors" depend also on the *other* coordinates in the system. Similarly, an *r*-coordinate vector is not the (purely spatial) radial direction for an arbitrary observer.

Conclusions

Radially moving observers, conveniently described using our generalisations of Gullstrand-Painleve and Lemaitre coordinates, determine interesting slicings of spacetime alternate to the static slicing inspired by Schwarzschild-Droste coordinates. Many familiar statements should require a qualifer "...according to the static slicing/observers" for clarity and accuracy. In future, I intend to study the quantum vacuum and Hawking radiation in these coordinates.

References:
- Finch (2015), *gr-qc/1211.4337*
- Gautreau & Hoffmann (1978)
- Martel & Poisson (2001), *gr-qc/0001069*
- Parikh & Wilczek (2000), *hep-th/9907001*
- Taylor & Wheeler, *Exploring Black Holes* (2000)