Thermal Stability and Quasi-stability Of Quantum Black Holes

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I. INTRODUCTION

It is well-known from semiclassical analyses that nonextremal, asymptotically flat black holes are thermally unstable due to decay under Hawking radiation, leading to their specific heat being negative [1]. This interesting fact has motivated the study of thermal stability of black holes, from a perspective that is inspired by a definite proposal for *quantum* spacetime (like Loop Quantum Gravity (LQG), [2, 3]) rather than on semiclassical assumptions. A consistent understanding of the issue of *quantum* black hole entropy has been obtained through LQG [4, 7], where not only has the Bekenstein-Hawking area law been retrieved for macroscopic (astrophysical) black holes, but a whole slew of corrections to it, due to quantum spacetime fluctuations have been derived as well [5]-[15], with the leading correction being logarithmic in area with the coefficient $-3/2$. However, we hasten to add the general disclaimer that *our paper is neither on LQG, nor does it use the LQG framework in an essential way !* LQG, if anything, plays only a motivational role in our work. Many of the assumptions of the paper, actually made independently of LQG, are justified on the ground that LQG might provide situations where these assumptions are valid. Of course, these are assumptions nevertheless. It would be most gratifying if these assumptions could be justified as results derived from LQG. But this is beyond our ability at this point.

The implications of this quantum perspective, on the thermal stability of black holes from decay due to Hawking radiation, have therefore been an important aspect of black hole thermodynamics beyond semi-classical analysis, and also somewhat beyond the strictly equilibrium configurations that Isolated Horizons represent. Classically a black hole in general relativity is characterized by its' mass (M), charge (Q) and angular momentum (J). Intuitively, therefore, we expect that thermal behaviour of black holes will depend on all of these parameters. For a given classical metric characterizing a black hole, the mass can be derived explicitly to be a function of the charge and angular momentum. However, the quantum spacetime perspective frees us from having to use classical formulae for this functional dependence of the mass. Instead, the assumption is simply this : the mass is a monotonically increasing function of the horizon *area*, alongwith the charge and angular momentum.

The simplest case of vanishing charge and angular momentum has been investigated longer than a decade ago [16] - [18]. This has been generalized, via the idea of *thermal* holography [19], [20], and the saddle point approximation to evaluate the canonical partition function corresponding to the horizon, retaining Gaussian thermal fluctuations. The consequence is a general criterion of thermal stability as an inequality connecting area derivatives of the mass and the microcanonical entropy. This inequality is nontrivial only when the microcanonical entropy has corrections (of a particular algebraic sign) beyond the area law, as is the case for the loop quantum gravity calculation of the microcanonical entropy [6].

This body of work has been generalized a few years ago [21] to include black holes with charge, and more recently, to radiant horizons which are both charged and rotating [29]-[30]. The complex set of stability criteria now decisively predict the thermal stability or otherwise, of any radiant black hole, when its mass-area relation extracted from its classical metric is used as input. The inclusion of rotation poses challenges in the LQG formulation [13], [9] - [11] of isolated horizons. However, the general understanding of non-radiant rotating isolated horizons has parallels in these assays. We do not review this body of work, but realize that the thermal stability behaviour of rotating radiant black holes may be *qualitatively different* from that of the

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non-rotating ones. Further, while investigating several types of black holes in diverse spacetime dimensions, considered as examples where the ensuing thermal stability criteria are applicable, the inclusion of rotation does throw up surprises : the existence of small but nontrivial windows in parameter space, where some black holes, considered unstable generically, appear to satisfy most of our criteria of thermal stability. However, one important thermal stability criterion is still violated, and in this respect, the standard wisdom regarding thermal instability of asymptotically flat black holes, appears to still hold. However, what has not been anticipated in earlier work is the existence of such narrow windows of 'quasi'-stability of asymptotically flat black holes.

II. THERMAL HOLOGRAPHY

In this section, we present a generalization of the thermal holography for non-rotating electrically charged quantum radiant horizons discussed in [21], to the situation when the horizon has both charge and angular momentum. Such a generalization completes the task set out in [16] and [19] to include charge and angular momentum in consideration of thermal stability of the horizon under Hawking radiation.

The Hilbert space of a generic quantum spacetime is given as, $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_v$, where $b(v)$ denotes the boundary (bulk) space. A generic quantum state is thus given as

$$|\Psi\rangle = \sum_{b,v} C_{b,v} |\chi_b\rangle \otimes |\psi_v\rangle \tag{1}$$

Now, the full Hamiltonian operator $(\widehat{H})$, operating on $\mathcal{H}$ is given by

$$\widehat{H}|\Psi\rangle = (\widehat{H_b} \otimes I_v + I_b \otimes \widehat{H_v})|\Psi\rangle \tag{2}$$

where, respectively, $I_b(I_v)$ are identity operators on $\mathcal{H}_b(\mathcal{H}_v)$ and $\widehat{H_b}(\widehat{H_v})$ are the Hamiltonian operators on $\mathcal{H}_b(\mathcal{H}_v)$.

The first class constraints are realized on Hilbert space as annihilation constraints on physical states. The bulk Hamiltonian operator thus annihilates bulk physical states

$$\widehat{H_v}|\psi_v\rangle = 0 \tag{3}$$

The bulk quantum spacetime is assumed to be free of electric charge and angular momentum, so that eqn. (3) is augmented by the relation

$$[\widehat{H_v} - \Phi\widehat{Q_v} - \Omega\widehat{J_v}]|\psi_v\rangle = 0 \ . \tag{4}$$

This assumption gleans from the idea that the generic quantum bulk Hilbert space is invariant under local $U(1)$ gauge transformations and local spacetime rotations (the latter, as part of local Lorentz invariance). Inertial frame dragging certainly does not detract from such invariances.

We now consider, heuristically, a grand canonical ensemble of quantum spacetimes with horizons as boundaries, in contact with a heat bath, at some (inverse) temperature β. The Grand Partition Function is given by

$$Z_G = Tr(exp(-\beta\widehat{H} + \beta\Phi\widehat{Q} + \beta\Omega\widehat{J})) \tag{5}$$

where the trace is taken over all states. This definition, together with eqn.s (1) and (4), yields

$$\begin{aligned} Z_G &= \sum_{b,v} |C_{b,v}|^2 \langle\psi_v|\psi_v\rangle\langle\chi_b|exp(-\beta\widehat{H} + \beta\Phi\widehat{Q} + \beta\Omega\widehat{J})|\chi_b\rangle \\ &= \sum_b |C_b|^2 \langle\chi_b|exp(-\beta\widehat{H} + \beta\Phi\widehat{Q} + \beta\Omega\widehat{J})|\chi_b\rangle \ , \end{aligned} \tag{6}$$

assuming that the bulk states are normalized. The partition function thus turns out to be completely determined by the boundary states (Z_{Gb}), i.e.,

$$\begin{aligned} Z = Z_{Gb} &= Tr_b \exp(-\beta\widehat{H} + \beta\Phi\widehat{Q} + \beta\Omega\widehat{J}) \\ &= \sum_{k,l,m} g(k,l,m) \quad \exp(-\beta(E(A_k, Q_l, J_m) - \Phi Q_l - \Omega J_m)) \ , \end{aligned} \tag{7}$$

where $g(k,l,m)$ is the degeneracy corresponding to energy $E(A_k, Q_l, J_m)$ and k, l, m are the quantum numbers corresponding to eigenvalues of area, charge and angular momentum respectively. These quantum numbers are all taken to be discrete [12]. Here, the spectrum of the boundary Hamiltonian operator is assumed to be a function of area, charge and angular momentum of the boundary, considered here to be the horizon. Following [8]-[12], it is further assumed that these 'quantum hairs' all have a discrete spectrum, in parallel with the LQG results. We cannot prove these assumptions at this point, but explore their consequences here. In the Macroscopic area limit ($A_h >> l_P^2$) of quantum isolated horizons , they all have large eigenvalues i.e. ($k, l, m >> 1$), so that, application of the Poisson resummation formula [16] gives

$$Z_G = \int dx\ dy\ dz\ g(A(x), Q(y), J(z))\ \exp(-\beta(E(A(x), Q(y), J(z)) - \Phi Q(y) - \Omega J(z))) \tag{8}$$

where x, y, z are respectively the continuum limit of k, l, m respectively.

We now assume that the macroscopic spectra of the area, charge and angular momentum are *linear* in their arguments, so that a change of variables gives, with constant Jacobian, the result

$$Z_G = \int dAdQdJ\ \exp[S(A) - \beta(E(A, Q, J) - \Phi Q - \Omega J)]\ , \tag{9}$$

where, following [22], the *microcanonical* entropy of the horizon is defined by $\exp S(A) \equiv \frac{g(A(x),Q(y),J(z))}{\frac{dA}{dx}\frac{dQ}{dy}\frac{dJ}{dz}}$.

III. STABILITY CRITERIA

Using the saddle point approximation to evaluate eqn (9), we obtain the thermal stability criteria that govern most radiant black holes

$$H = \begin{pmatrix} \beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}) & \beta M_{AQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{AJ}(\bar{A}, \bar{Q}, \bar{J}) \\ \beta M_{AQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{JQ}(\bar{A}, \bar{Q}, \bar{J}) \\ \beta M_{AJ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{JQ}(\bar{A}, \bar{Q}, \bar{J}) & \beta M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) \end{pmatrix} \tag{10}$$

The necessary and sufficient conditions for a real symmetric square matrix to be positive definite are : 'determinants all principal square submatrices, and the determinant of the full matrix, are positive.'[23] This condition leads to the following 'stability criteria' :

$$\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}) > 0 \tag{11}$$

$$\beta M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) > 0 \tag{12}$$

$$\beta M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) > 0 \tag{13}$$

$$M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) - (M_{JQ}(\bar{A}, \bar{Q}, \bar{J}))^2 > 0 \tag{14}$$

$$M_{JJ}(\bar{A}, \bar{Q}, \bar{J})(\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A})) - \beta(M_{AJ}(\bar{A}, \bar{Q}, \bar{J}))^2 > 0 \tag{15}$$

$$M_{QQ}(\bar{A}, \bar{Q}, \bar{J})(\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A})) - \beta(M_{AQ}(\bar{A}, \bar{Q}, \bar{J}))^2 > 0 \tag{16}$$

$$\begin{aligned} &[(\beta M_{AA}(\bar{A}, \bar{Q}, \bar{J}) - S_{AA}(\bar{A}))(M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) - (M_{JQ}(\bar{A}, \bar{Q}, \bar{J}))^2) \\ &- \beta M_{AQ}(\bar{A}, \bar{Q}, \bar{J})(M_{AQ}(\bar{A}, \bar{Q}, \bar{J}) M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) - M_{JQ}(\bar{A}, \bar{Q}, \bar{J}) M_{AJ}(\bar{A}, \bar{Q}, \bar{J})) \\ &\quad + \beta M_{AJ}(\bar{A}, \bar{Q}, \bar{J})(M_{AQ}(\bar{A}, \bar{Q}, \bar{J}) M_{JQ}(\bar{A}, \bar{Q}, \bar{J}) - M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) M_{AJ}(\bar{A}, \bar{Q}, \bar{J}))] > 0 \end{aligned} \tag{17}$$

Of course, (inverse) temperature β is assumed to be positive for a stable configuration. This makes the inequalities 12, 13 simpler i.e. $M_{QQ}(\bar{A}, \bar{Q}, \bar{J}) > 0, M_{JJ}(\bar{A}, \bar{Q}, \bar{J}) > 0$.

Now, the temperature is defined as $T \equiv \frac{1}{\beta}$; eqn. (11) implies that $T = \frac{M_A}{S_A}$. Eqn.s (18) and (19) together yield $S_A = \frac{1}{4A_P} - \frac{3}{2A}$ and is positive for macroscopic black holes as $A >> A_P$. So, positivity of M_A implies the positivity of β for macroscopic black holes. The relation $T = \frac{M_A}{S_A}$ implies $\frac{dT}{dA} = \frac{M_A}{(S_A)^2}(\beta M_{AA} - S_{AA})$. So, what is new is the requirement that this temperature must increase with horizon area, inherent in the positivity of the quantity $(\beta M_{AA} - S_{AA})$ which appears in several of the stability criteria. If this is violated, as for example in case of the standard Schwarzschild black hole (reference), thermal instability is inevitable.

The convexity property of the entropy follows from the condition of convergence of partition function under gaussian fluctuations [16], [22], [25]. The thermal stability is related to the convexity property of entropy. Hence, the above conditions are correctly the conditions for thermal stability. For chargeless, non-rotating horizons, eqn. (11) reproduces the thermal stability criterion and condition of positive specific heat(i.e.variation of black hole mass with temperature) given in [19], as expected. Actually for a chargeless, non-rotating black hole, both the mass and the temperature are functions only of the horizon area A. From these one can define the specific heat as $C = \frac{dM}{dT} = \frac{(S_A)^2}{(\beta M_{AA} - S_{AA})}$.

For charged, non-rotating black holes, eqn.s (11), (12) and (16) describe the stability, in perfect agreement with [21], while (11), (13) and (15) describe the thermal stability criteria for uncharged rotating radiant horizons. The new feature for black holes with both charge and angular momentum is that not only does the specific heat has to be positive for stability, but the charge and the angular momentum play important roles as well.

As claimed in the Introduction, the thermal stability criteria above are derived by the application of standard statistical mechanical formalism to a quantum horizon characterized by various observables having discrete eigenvalue spectra. Thus, no aspect of classical geometry enters the derivation of these criteria. Given the classical metrics specifying various classical black hole spacetimes, the mass can be obtained as an explicit function of the area, charge and angular momentum of the horizon. It is then possible, on the basis of our stability criteria, to *predict* which classical black holes will radiate away to extinction, and which ones might find some stability, and for what range of parameters. This is what is attempted in the next section.

IV. PREDICTING THERMAL STABILITY AND QUASI-STABILITY

Notice that in the stability criteria derived in the last section, first and second order derivatives of the microcanonical entropy of the horizon at equilibrium play a crucial role, in making some of the criteria non-trivial. Thus, corrections to the microcanonical entropy beyond the Bekenstein-Hawking area law, arising due to quantum spacetime fluctuations might play a role of some significance. It has been shown that [6] the microcanonical entropy for *macroscopic* isolated horizons has the form

$$S = S_{BH} - \frac{3}{2} \log S_{BH} + \mathcal{O}(S_{BH}^{-1}) \tag{18}$$

$$S_{BH} = \frac{A_h}{4A_P} \ , \ A_P \equiv \text{Planck area} \ . \tag{19}$$

A. Kerr-Newman Black Hole

The Kerr-Newman metric of asymptotically flat Black Hole is given in Boyer-Lindquist coordinates as

$$ds^2 = -\frac{\Xi}{\Sigma}(dt - a\, sin^2\theta\, d\phi)^2 + \frac{sin^2\theta}{\Sigma}((r^2+a^2)d\phi - adt)^2 + \frac{\Sigma}{\Xi}dr^2 + \Sigma d\theta^2 \tag{20}$$

where, $\Xi = r^2 - 2\,M\,r + a^2 + Q^2, \quad \Sigma = r^2 + a^2\,cos^2\theta, \quad a = \frac{J}{M}$ The generalized Smarr formula for the Kerr-Newman Black Hole is given as [26]

$$M^2 = \frac{A}{16\pi} + \frac{\pi}{A}(4J^2 + Q^4) + \frac{Q^2}{2} \tag{21}$$

We now define the inverse Hawking temperature $\beta \equiv (S_A/M_A)$ and logarithmic rate of change of the temperature with horizon area $\Delta \equiv (S_A/M_A)M_{AA} - S_{AA}$, both quantities being defined at thermal equilibrium. We obtain

$$\begin{aligned} \beta &> 0 \text{ for } A^2 > 16\pi^2(4J^2+Q^4) \\ \Delta &> 0 \text{ for } 96\pi^2(4J^2+Q^4) > A^2 > 16\pi^2(4J^2+Q^4) \end{aligned} \tag{22}$$

The second inequality in (22) is unexpected, since most earlier analyses had implied that there is no such window of (quasi)-stability for asymptotically flat black hole spacetimes, as, for instance. $\Delta < 0$ for the standard Schwarzschild solution. The issue though is that whether such a window applies vis-a-vis the other stability criteria.

We find that, for the Kerr-Newman spacetime,

$$\begin{aligned} m_j &\equiv \Delta M_{JJ} - \beta(M_{AJ})^2 \\ &\propto \frac{8\pi^3\left(-16J^4+8J^2Q^4+3Q^8\right)}{A^5} + \frac{16\pi^2Q^6}{A^4} + \frac{-4J^2\pi+3\pi Q^4}{A^3} - \frac{1}{32\pi A} \\ &= \frac{8\pi^3(4J^2+Q^4)(3Q^4-4J^2)}{A^5} + \frac{16\pi^2Q^6}{A^4} + \frac{-4J^2\pi+3\pi Q^4}{A^3} - \frac{1}{32\pi A} \ . \end{aligned} \tag{23}$$

The proportionality constant is of course a positive quantity. It is clear that if $3Q^4 > 4J^2$ by a sufficient margin, then it is possible to make the entire $m_j > 0$ where A^2 lies within the range given by (22). E.g., if $Q^4 = 4J^2$, then $m_j > 0$ for $A^2 < 64\pi^2Q^4$ which lies within the range (22).

Similarly,

$$\begin{aligned} m_q &\equiv \Delta M_{QQ} - \beta(M_{AQ})^2 \\ &\propto \frac{4\pi^3Q^2\left(4J^2+Q^4\right)\left(36J^2+Q^4\right)}{A^5} \\ &+ \frac{\pi^2\left(48J^4+88J^2Q^4+3Q^8\right)}{A^4} + \frac{\pi Q^2\left(36J^2+Q^4\right)}{2A^3} \\ &+ \frac{\frac{3J^2}{2}-\frac{Q^4}{8}}{A^2} - \frac{3Q^2}{64\pi A} - \frac{1}{256\pi^2} \ . \end{aligned} \tag{24}$$

A similar argument for m_q, as for m_j, shows that $m_q > 0$ within the range (22), provided, $q^4 < 12j^2$. This is also true for the inequality

$$m_{jq} \equiv M_{QQ}M_{JJ} - (M_{QJ})^2 \tag{25}$$

as a simple calculation quickly reveals. It follows that the 'quasi-stability' window is indeed valid for eqn.s (11)=(16); the question is : does it hold for the determinant of the full Hessian matrix, i.e., criterion (17) ? We find that this determinant is

$$\begin{aligned} H_{KN} \propto\ & \frac{4\pi^3Q^2\left(-48J^4-8J^2Q^4+Q^8\right)}{A^5} - \frac{\pi^2\left(16J^4+24J^2Q^4-3Q^8\right)}{A^4} \\ +\ & \frac{\pi Q^2\left(-12J^2+Q^4\right)}{2A^3} - \frac{\left(4J^2+Q^4\right)}{8A^2} - \frac{3Q^2}{64\pi A} - \frac{1}{256\pi^2} \end{aligned} \tag{26}$$

A close examination reveals that H_{KN} *cannot* be positive in the region (22) ! This implies that the window of 'quasi'-stability (22) cannot be promoted to a full-fledged window of stability, even though all but one of the stability criteria are fulfilled within this window of parameter space. The actual reason as to why this happens is not clear to us at this moment. Be that as it may, the final conclusion tallies with the general wisdom that asymptotically flat black holes tend to be thermally unstable.

V. SUMMARY AND DISCUSSION

The most important phyical upshot in our stability analysis, with the inclusion of rotation, is the existence of windows in parameter space of black holes where most of the thermal stability criteria are apparently satisfied, for *asymptotically flat* spacetimes, even though the Hessian determinant condition of thermal stability is violated. This is somewhat unexpected, as it departs from the wisdom of conventional semiclassical thermodynamic analyses of specific black holes [24]. Such windows of 'quasi'-stability in largely unstable black hole spacetimes have connotations for Hawking radiation which warrant further analyses. Clearly, inclusion of rotation and charge appear to add a degree of stability against Hawking decay of black holes, but apparently not enough to render them completely stable.

We reiterate that our analysis of stability citeria is quite independent of specific classical spacetime geometries, relying as it does on quantum aspects of spacetime. The construction of the partition function used standard formulations of equilibrium statistical mechanics augmented by results from canonical Quantum Gravity, with extra inputs regarding the behaviour of the microcanonical entropy as a function of area *beyond the Bekenstein-Hawking area law*, as for instance derived from Loop Quantum Gravity [6]. However, we emphasize that the results are more general than being restricted to any specific proposal for quantum spacetime geometry, requiring only certain functional dependences on horizon area and other parameters of statistical mechanical quantities like entropy. It also stands to reason that our stability criteria are useful for predicting the thermal behaviour vis-a-vis Hawking radiation for specific astrophysical black holes. In particular, our criteria precisely predict regions of the parameter space of specific black hole solutions, not only in general relativity, but also of extensions inspired from warped geometries and string theories, where these solutions are stable under Hawking radiation.

It is also noteworthy that the approach is useful for making predictions on the thermal stability of black holes in Lorentzian spacetimes with arbitrary number of spatial dimensions. It can also be generalized to black holes with arbitrary 'hairs' (charges) - either quantum or classical [30].

There are however, subtleties of a statistical mechanical nature which have not been addressed in this paper. The most important of these is the *nature* of the thermal instability discerned by us. While there are indications that the instability in most cases can be associated with some sort of phase transition [19], the very general approach here has not yet been applied to discuss the full range of thermal behaviour exhibited specifically for AdS Schwarzschild black holes, for instance, as discussed in detail in [24]. Crucially, there are 'phases' discussed in that paper which have not been fully explored via our more 'quantum geometry' approach, as distinct from the semiclassical approach employed in [24]. We hope to return to these important issues in a future publication.

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