Gravitational stability analysis in post-Newtonian gravity

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The relativistic version of the gravitational instability criterion is studied in the first order of the post-Newtonian limit. In other words, we investigate the post-Newtonian Jeans analysis for a self-gravitating system whose characteristic velocity and corresponding gravitational field limit to the post- Newtonian regime. We find the first relativistic corrections to the jeans mass by obtaining the post-Newtonian corrections of dispersion relation of small perturbations which propagate in the post-Newtonian fluid. We have shown that the post-Newtonian Jeans mass can be smaller than the standard Jeans mass. In fact, this study reveals that the pressure and internal energy can have the gravitational effects even in the first post-Newtonian limit. Therefore, the post-Newtonian system can be more unstable by increasing the pressure and internal energy. Finally we have calculated the new Jeans mass for several different astrophysical systems in which temperature/pressure is very high. By adding the PN corrections in these systems, we have shown that these physical systems can be threatened by gravitational instabilities in PN gravity.

Keywords: Jeans analysis, post-Newtonian approximation, and gravitational instability

1. Introduction

The gravitational collapse occurs when the gravitational force is stronger than the internal gas pressure in non-rotating environment. The gravitational instability criterion/Jeans instability expresses the condition of this type of instability. In addition to Newtonian gravity, the Jeans instability has been investigated in the some gravitational theories. For example see Ref. 1–4. In this paper, we study the relativistic generalization of the Jeans instability in post-Newtonian (PN) gravity. We find the stability criterion for a self-gravitating system whose characteristic velocity and corresponding gravitational field are higher than a non-relativistic case and limit to the PN regime. We apply the results to astrophysical systems, e.g., hyper massive neutron stars, HMNSs, and neutrino dominated accretion flows, NDAFs, and show that there is a significant difference with the non-relativistic analysis.

This paper is organized as following. In Sec. 2, the dispersion relation is derived in PN gravity. In Sec. 3, we find the PN Jeans mass and in Sec. 4, we apply it to some high temperature systems. Finally in Sec. 5, results are discussed.

2. Gravitational instability criterion in the post-Newtonian limit

In this section we apply the PN hydrodynamics. Ref. 5,6. The matter variables in the PN method are $\{\rho^*, p, \Pi, \boldsymbol{v}\}$ where $\rho^* = \sqrt{-g}\gamma\rho$ is the conserved mass density in which ρ is the proper density and g is the determinant of the metric tensor. p is the pressure, $\Pi = \epsilon/\rho^*$ is the internal energy per unit mass, ϵ is the proper internal energy density and $\boldsymbol{v}$ is the fluid's velocity field. It should be mentioned that we study a perfect fluid throughout this paper. In fact, PN hydrodynamic equations which are completely derived in Ref. 5,6 and an equation of state can determine the behavior of a PN perfect fluid. To find the linear perturbation growth in PN approximation, we linearize the PN hydrodynamic equations and solve them. To do so, we perturb quantities in these relations and replace the Fourier expansion as $Q_1 = Q_a e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ for all perturbed quantities. After some manipulations, and expanding the solution up to $O(c^{-2})$, we obtain the following PN dispersion relation which is governing the propagating of the first order perturbations.

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0^* - \frac{1}{c^2}\Big\{\Big(\Pi_0 + \frac{p_0}{\rho_0^*}\Big)\Big(c_s^2 k^2 + 4\pi G\rho_0^*\Big) + \frac{32\pi^2 G^2 \rho_0^{*2}}{k^2}\Big\} + O(c^{-4}) \quad (1)$$

where $c_{\rm s}$ is the sound speed. Here, we assume that the equation of state is given by $p = p(\rho^*)$ and the background fluid is static, infinite and homogeneous. By discarding the PN terms, the standard dispersion relation is recovered. Equation (1) is the main result of this section that helps to extract a relativistic criterion for stability of the PN system against small perturbation.

3. Post-Newtonian Jeans mass

In this section, by using equation (1), we obtain the PN version of Jeans mass. Let us introduce the following parameters in order to simplify this relation.

$$c_{\rm sp}^2 = (1-\alpha)c_{\rm s}^2 \quad , \quad G_{\rm p} = (1+\alpha)G \quad (2)$$

where $\alpha = (p_0/\rho_0^* + \Pi_0)/c^2$, $c_{\rm sp}$ [a] is an effective sound speed and $G_{\rm p}$ is an effective gravitational constant. As one can see, the PN corrections decrease the sound velocity and increase the gravitational constant. By inserting these new parameters into equation (1), we have

$$\omega^2 = c_{\rm sp}^2 k^2 - 4\pi G_{\rm p}\rho_0 - \frac{32\pi^2 G^2 \rho_0^2}{k^2 c^2} \quad (3)$$

Now, setting $\omega = 0$ in above equation, we can find the gravitational instability criterion to $O(c^{-2})$ in the PN approximation. In this case, the PN Jeans wavenumber which has the PN correction terms is

$$k_{\rm Jp}^2 \simeq k_{\rm J}^2\Big(1 + \frac{2}{c^2}\big(c_{\rm s}^2 + \frac{p_0}{\rho_0} + \Pi_0\big)\Big) \quad (4)$$

in which the standard Jeans wavenumber $k_{\rm J}$ is defined as $k_{\rm J}^2 = 4\pi G\rho_0/c_s^2$. In this analysis a system is stable if wavenumber of the perturbation, k, is more than $k_{\rm Jp}$. This equation obviously shows that $k_{\rm Jp} > k_{\rm J}$.

Here, we find the Jeans mass $m_{\rm Jp}$ and the Jeans mass-energy $M_{\rm Jp}$ in the PN limit. $m_{\rm Jp}$ and $M_{\rm Jp}$ are constructed from matter density ρ_0 and matter-energy

[a] The p index represents the constants in PN theory and we use this notation in the rest of this paper.

density $\epsilon_0 = (1 + \Pi_0/c^2)\rho_0$ respectively. As we know, the Jeans mass is a mass inside a sphere with diameter λ. So, we have

$$m_{\rm Jp} \simeq m_{\rm J}\Big(1 - \frac{3}{c^2}\big(c_{\rm s}^2 + \frac{p_0}{\rho_0} + \Pi_0\big)\Big) \quad , \quad M_{\rm Jp} \simeq m_{\rm J}\Big(1 - \frac{3}{c^2}\big(c_{\rm s}^2 + \frac{p_0}{\rho_0} + \frac{2}{3}\Pi_0\big)\Big) \qquad (5)$$

in which $m_{\rm J}$ is the standard Jeans mass given by $m_{\rm J} = \pi\rho_0\lambda_{\rm J}^3/6$. These equations reveal that the PN Jeans mass and mass-energy are smaller than the standard case. In fact, increasing the sound speed, p_0/ρ_0, and internal energy reduce the Jean mass in PN gravity. Here, we show that pressure and internal energy can produce gravity even in the 1PN approximation. We apply the ideal-fluid equation of state to obtain PN Jeans mass as a function of temperature and also to simplify the analysis, we introduce the dimensionless temperature $\theta = \frac{k_B T}{\mu m_H c^2}$ and the dimensionless Jeans mass $\mathfrak{m}_{\rm J} = (\gamma\,\theta)^{3/2}$. Then we plot the Newtonian and PN Jeans masses in terms of θ in Fig.1[b]. This figure shows that at low temperatures θ, there is not any difference

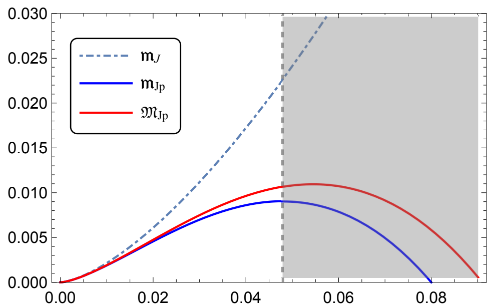

Fig. 1. The PN dimensionless Jeans mass $\mathfrak{m}_{\rm Jp}$ and Jeans mass-energy $\mathfrak{M}_{\rm Jp}$ in terms of dimensionless temperature θ. The vertical dashed line corresponds to $\theta_{\rm c} = 0.048$.

between $\mathfrak{m}_{\rm Jp}$ and $\mathfrak{m}_{\rm J}$, and after $\theta \sim 0.01$ the difference increases. It also expresses that $\mathfrak{m}_{\rm Jp}$ and $\mathfrak{M}_{\rm Jp}$ have maximums at $\theta = 0.048$ and $\theta = 0.055$, respectively. In fact, after these high temperatures our system is not in the 1 PN regime and we should consider other PN corrections. So, allow us to take $\theta_{\rm c} = 0.048$ as a critical temperature where our analysis does not work. Therefore, our analysis is limited to $\theta < \theta_{\rm c}$. We exhibit this boundary with vertical line in Fig. 1.

4. Application to high temperature environments

In the this section we derive the gravitational instability criterion of some high temperature astrophysical systems in the PN approximation and compare it with the Newtonian case. We have introduced some of the physical properties of these systems in the Table 1.

[b]Without loss of generality, we assume that the background fluid is monoatomic, i.e., $\gamma = 5/3$ and evolves adiabatically.

Table 1. Characteristics of the various astrophysical systems

System	$n\,(\mathrm{cm}^{-3})$	$T\,(\mathrm{K})$	Size (pc)[a]	$\lambda_{\mathrm{Jp}}\,(\mathrm{pc})$	$100\times\Delta m_{\mathrm{J}}/m_{\mathrm{J}}$
ICM	10^{-3}	$10^{7}-10^{8}$	$\sim 10^{6}$	$(3.28-10.37)\times 10^{6}$	$(1.86-18.63)\times 10^{-3}$
Fermi Bubble	10^{-2}	$10^{8}-10^{9}$	10^{4}	$(3.28-10.36)\times 10^{6}$	$(1.86-18.63)\times 10^{-2}$
HMNS	10^{39}	$10^{10}-10^{11}$	$\sim 10^{-13}$	$(1.03-3.07)\times 10^{-13}$	$(1.86-18.63)$
NDAF	10^{37}	10^{11}	$\sim 10^{-11}$	30.75×10^{-11}	18.63

Note: [a] n and T are the number density and temperature respectively. We have set $\mu = 0.615$

Table 1 clearly expresses that for systems at very high temperatures (i.e., $T \gtrsim 10^{11}$K) m_{J} and m_{Jp} have significant difference. Except for the rest of the cases, this difference is about 19% for HMNS and NDAF where temperature is very high. It should be mentioned that the PN Jeans wavelength is of the same order of magnitude as the characteristic size of these two cases. For the last one, the dynamical time scale known as the accretion time (for more detail see Ref. 7) of this model at $r = 4r_{\mathrm{s}}$, for black hole mass $M = 3M_{\odot}$, and viscosity parameter $\alpha = 0.1$ is estimated about $\sim 1.3\times 10^{-2}$s which is greater than $t_{\mathrm{ff}} \sim 10^{-3}$s. Accordingly, this system can be locally unstable. We have also considered two cases with two different central black holes of this model and exhibited the results in the Table 2.

Table 2. Properties of NDAFs at different radii.

M_{BH}	$r(r_s)$	$n(10^{36}\mathrm{cm}^{-3})$	$\mathrm{T}(10^{10}\mathrm{K})$	$m_{\mathrm{J}}(M_{\odot})$	$\lambda_{\mathrm{J}}(km)$	$\Delta m_{\mathrm{J}}/m_{\mathrm{J}}^{\mathrm{a}}$
	4	24.44	12.73	1.99	64.11	0.21
$5M_{\odot}$	10	7.52	5.80	1.10	78.02	0.09
	40	1.27	1.77	0.45	105	0.03
	4	32.89	15.52	2.30	61.02	0.25
$10M_{\odot}$	10	10.13	7.07	1.28	74.25	0.11
	40	1.70	2.15	0.52	99.94	0.04

Note: [a] We have assumed that $\mu = 0.70$.

This table clearly reveals that at smaller radii where the temperature of the system is higher, the fractional difference is larger. This means that such a system can be locally unstable at internal radius in the 1PN approximation. As we know, this fact can effectively change the dynamics of the accretion.

5. Conclusion

We have shown that the PN Jeans mass is smaller than the standard one at 1 PN order. We have also concluded that pressure and internal energy can in principle have the relativistic effects and play the role of the gravitational force even in $O(c^{-2})$. We have also derived the PN Jeans mass for some astrophysical systems in which temperature is very high. We have deduced that some of these physical systems can be seriously threatened by gravitational instabilities in PN theory.

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