Gravitational waves from spinning binary black holes at the leading post-Newtonian orders at all orders in spin

Nils Siemonsen[1*], Jan Steinhoff[2] and Justin Vines[2]

[1] *Department of Physics, Eidgenössische Technische Hochschule Zürich Otto-Stern-Weg 1, 8093 Zürich, Switzerland*

[2] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute) Am Mühlberg 1, 14476 Potsdam, Germany*

[*] *E-mail: sinils@student.ethz.ch*

We determine the binding energy, the total gravitational wave energy flux, and the gravitational wave modes for a binary of rapidly spinning black holes, working in linearized gravity and at leading orders in the orbital velocity, but to all orders in the black holes' spins. Though the spins are treated nonperturbatively, surprisingly, the binding energy and the flux are given by simple analytical expressions which are finite (respectively third- and fifth-order) polynomials in the spins. Our final results are restricted to the important case of quasi-circular orbits with the black holes' spins aligned with the orbital angular momentum.

Keywords: Binary black holes; Gravitational waves; nonperturbative method

1. Introduction

The general relativistic two-body problem, in particular the description of compact binaries, as one of the most important sources of gravitational waves (GW) detectable from Earth, poses a large challenge for analytical as well as numerical calculations. Accurate waveform models are an essential ingredient for the detection of GW signals. The detections of such signals [1] have demonstrated the success of the approaches, but the need for more accurate and more general descriptions of binary dynamics in general relativity (GR) persists and will grow with future more sensitive GW detectors. Several (perturbative) analytic approaches to the two-body problem in GR have been developed. Among others, the post-Newtonian (PN) expansion. It is an analytic approach for arbitrary mass ratio compact binaries in the weak-field, slow-motion (characterized by $\epsilon_{\rm PN} \sim v^2/c^2 \sim Gm/rc^2$) and small spin regime (determined by the expansion parameter $\epsilon_{\rm spin} \sim Gm\chi/rc^2$)[a].

Previously, PN results for binary black holes (BBHs), in particular for the black holes' (BH) spins, were obtained order by order in the perturbative series, since for BBHs: $\epsilon_{\rm PN} \sim \epsilon_{\rm spin}$. However, by treating both expansion parameters separately, we are able to derive compact analytic expressions for gauge invariant quantities (GW energy flux, GW modes, conserved energy etc.) for a BBH at the leading PN orders to all orders in the BHs' spins – treating the spins nonperturbatively.

[a] With $cS/Gm^2 = \chi \in [0, 1)$, the dimensionless spin parameter.

2. Effective Action

In a harmonic gauge framework, with a linearized metric perturbation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \sim \mathcal{O}(G)$, following the derivations in Ref. 2 and 3, we implement an effective action of a two point-particle system with a spin-induced multipole structure (for each particle) and respective multipole couplings to the gravitational field. In $G = c = 1$ units, this effective description is split into particle- and spin-kinetic terms $S_{\rm kin}[\mathbf{T}_A]$, interaction terms $S_{\rm int}[\mathbf{h}, \mathbf{T}_A]$ of the Ath black hole and a term, $S_G[\mathbf{h}]$, containing the gravitational field's dynamics at linear order in $h_{\mu\nu}$, so that

$$S^{\rm BBH}_{\rm eff}[\mathbf{h}, \mathbf{T}_1, \mathbf{T}_2] = S_G[\mathbf{h}] + \{S_{\rm kin}[\mathbf{T}_1] + S_{\rm int}[\mathbf{h}, \mathbf{T}_1] + (1 \leftrightarrow 2)\}.$$

All spin-induced multipoles of the black holes are coupling to the linear metric perturbation $h_{\mu\nu}$ by (still containing nonlinear velocity contributions)

$$S_{\rm int}[\mathbf{h}, \mathbf{T}_A] = \int dt \left\{ \sum_{\ell=0}^{\infty} \frac{m U^\mu_A U^\nu_A}{2\gamma_A\, \ell!} \mathrm{Re}\left[i^\ell a^L_A \partial_L h_{\mu\nu} + i^{\ell-1} a^\rho_A \epsilon_{\nu\rho}{}^{\alpha\beta} a^{L-1}_A \partial_\alpha \partial_{L-1} h_{\beta\mu} \right] \right\},$$

where a^μ_A is the spin 4-vector, U^μ_A the velocity 4-vector (parameterized by coordinate time t) and γ_A the Lorentz factor of the Ath BH respectively. $\mathbf{h}$ encompasses the gravitational degrees of freedom and $\mathbf{T}$ encodes the multipolar degrees of freedom of the individual black holes. Note, we used the multi-index notation $L := \mu_1 \ldots \mu_\ell$. This interaction term arises from considering all possible combinations of the vacuum Riemann tensor, multipole moments and particle's velocities at linear order in the metric perturbation, under the restriction of reparameterization invariance and invariance[4].

3. Conservative dynamics

From the above action principle, the linearized field equations in the near zone of the binary can be deduced (to linear order the velocities) and solved to all orders in spin. A Fokker-type action with Lagrangian (including all spin-spin couplings in this linearized framework with up to linear velocity contributions)

$$\begin{aligned} \mathcal{L} = \Big[-m_1 + \frac{m_1}{2} v_1^2 + \frac{1}{2} \boldsymbol{S}_1 (\boldsymbol{v}_1 \times \dot{\boldsymbol{v}}_1) + \boldsymbol{S}_1 \cdot \bar{\boldsymbol{\Omega}}_1 + (1 \leftrightarrow 2) \Big] \\ + m_1 m_2 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!} \left[a_0^{2L} + \frac{2 v^i a_0^j \epsilon_{ij}{}^k a_0^{2L}}{(2\ell+1)} \frac{\partial}{\partial r^k} \right] \frac{\partial r^{-1}}{\partial r^{2L}}. \end{aligned} \tag{1}$$

in the center-of-mass frame, is obtained. We introduced the BHs' masses m_A, 3-velocities $\boldsymbol{v}_A(t)$, spin 3-vectors $\boldsymbol{a}_A = \boldsymbol{S}_A/m_A$ and combination $\boldsymbol{a}_0 = \boldsymbol{a}_1 + \boldsymbol{a}_2$, as well as the BHs' separation vector $\boldsymbol{r}$ with $r = |\boldsymbol{r}|$ and the angular velocity vector $\bar{\boldsymbol{\Omega}}_A$. The equations of motion for the BBH, in a quasi-circular limit and for spin vectors aligned with the orbital angular momentum of the system, can be resummed and solved for the angular frequency ω of the orbits in compact fashion. To linear order in the metric perturbation $h_{\mu\nu}$ and velocities $\boldsymbol{v}$ (i.e., at leading PN order), but to all orders in the spins $\boldsymbol{a}_A$, the angular frequency is given by $\ddot{\boldsymbol{r}} = -\omega^2 \boldsymbol{r}$, with $\omega^2 = M/r[r - v(2a_0 + \sigma^*)](r^2 - a_0^2)^{-3/2}$. Here we defined $M = m_1 + m_2$ and $\sigma^* = |\boldsymbol{\sigma}^*| = |m_2 \boldsymbol{a}_1 + m_1 \boldsymbol{a}_2|/M$.

Gauge invariant quantities (e.g., Noether charges or the GW modes) are conveniently expanded in the commonly used expansion parameter $x = (M\omega)^{2/3}$, such that $\epsilon_{\rm PN} \sim x$ and $\epsilon_{\rm spin} \sim x\chi$. In the following χ serves as book keeping parameter for the order in spin considered (e.g., $a_A \sim \mathcal{O}(\chi)$, $a_A^2 \sim \mathcal{O}(\chi^2)$, $\ldots$). At each order in χ, we consider the lowest order in x, i.e., the leading PN order at each order in spin. Notice that this is different from the traditional PN order counting in $1/c^2$ (since we treated $\epsilon_{\rm spin}$ and $\epsilon_{\rm PN}$ as independent).

The conserved energy and total angular momentum of the two-body system can be obtained as the associated Noether charges of (1). Recasting the angular frequency ω in x yields, together with the respective Noether charges as functions of r, the binding energy $E(x)$ and total angular momentum $J(x) = L(x) + m_1 a_1 + m_2 a_2$ perpendicular to the orbital plane of the BBH at the leading PN orders at all orders in spin

$$E(x) = -\frac{\mu x}{2}\left\{1 + \frac{x^{3/2}}{3M}(7a_0 + \delta a_-) - \frac{x^2 a_0^2}{M^2} - \frac{x^{7/2} a_0^2}{M^3}(a_0 - \delta a_-)\right\}, \text{ and}$$
$$L(x) = \mu x^{-1/2}\left\{M - \frac{5}{12}x^{3/2}(7a_0 + \delta a_-) + \frac{x^2 a_0^2}{M} + \frac{3x^{7/2}}{4M^2}a_0^2(a_0 - \delta a_-)\right\}.$$

Where the symmetric mass ratio $\nu = \mu/M$, the antisymmetric mass ratio $\delta = (m_1 - m_2)/M$, and a_- being the projection of $\boldsymbol{a}_- = \boldsymbol{a}_1 - \boldsymbol{a}_2$ orthogonal to the orbital plane.

Remarkably, the spin-expansions of the binding energy E and orbital angular momentum L terminate after cubic-in-spin contributions. Note that this polynomial structure in spin hinges on the use of x (or ω) as the variable in the conserved quantities.

4. Radiative sector

The far zone dynamics, i.e., the gravitational effects at future null infinity, are related with the near zone dynamics through a matching of the PN solution obtained above and the far zone post-Minkowskian expansion [5]. The matching procedure yields a relation between the source' multipole moments and the emitted GW modes and total GW energy flux at future null infinity. The GW polarization waveform $h_+ - ih_\times = \sum_{\ell,m} {}_{-2}Y_{\ell m} h_{\ell m}$ is projected onto a basis of spin weighted spherical harmonics ${}_{-s}Y_{\ell m}$. The GW modes $h_{\ell m}$ are explicitly given in eq. (84) of Ref. 3. The spin expansion of the even-m modes terminates at a finite order. The odd-m modes have contributions at all orders in spin, though they, are resumed in a compact form, i.e., terms like $\sqrt{M^2 + x^2 a_0^2}$ appear. Expanding these term, $\sqrt{M^2 + x^2 a_0^2} = M + x^2 a_0^2/(2M) - x^4 a_0^4/(8M^3) + \mathcal{O}(x^6\chi^6)$, yields the contributions at the leading PN order at each order in spin.

In an adiabatic approximation, at the leading PN orders at each order in the BHs' spin, we find the total GW energy flux

$$\begin{aligned}\mathcal{F} = \frac{\mu^2 x^5}{M^2}\Bigg[\frac{32}{5} - \frac{8x^{3/2}}{5M}\Big\{8a_0 + 3\delta a_-\Big\} + \frac{2x^2}{5M^2}\Big\{32a_0^2 + a_-^2\Big\} \\ - \frac{4x^{7/2}}{15M^3}\Big\{16a_0^3 + 2a_0a_-^2 + 52\delta a_0^2 a_- + \delta a_-^3\Big\} + \frac{2x^4 a_0^2}{5M^4}\Big\{16a_0^2 + a_-^2\Big\} \\ + \frac{2a_0^2 x^{11/2}}{15M^5}\Big\{64a_0^3 + a_0a_-^2 - 68\delta a_0^2 a_- - 3\delta a_-^3\Big\}\Bigg].\end{aligned}$$

Again, the infinite sets of spin-induced multipolar interactions of the two BHs remarkably cancel out at higher than quintic-in-spin contributions. Hence, the total energy flux conveys full information about the spin effects at leading PN order in the first five terms of the spin expansion.

5. Conclusion

We determined the binding energy, the gravitational wave modes and total energy flux emitted by a spinning nonprecessing binary black hole in quasi-circular motion at leading post-Newtonian orders at all orders in spin. Our results include contributions of arbitrarily large PN order, counting in $1/c^2$. In particular, we obtained for the first time the quartic-in-spin contributions to the 4PN waveform and total energy flux, along with all higher-order-in-spin contributions at the corresponding leading PN orders. Remarkably, the binding energy, the total energy flux, as well as the even-in-m gravitational wave modes only contain a finite number of nonzero contributions in their spin expansions at leading post-Newtonian order.

Conversely, the modes where all powers in spin appear are nevertheless rather compact, which can be used to improve the resummation of modes, e.g., in the synergetic EOB waveform model[6]. Though our results are only valid for aligned spins, they can still be used to approximate waveforms from precessing binaries[7].

References

1. B. P. Abbott *et. al.* (Virgo, LIGO Scientific), *Phys. Rev. Lett.* **116**, 24113 (2016).
2. J. Vines and J. Steinhoff, *Phys. Rev. D* **97**, 064010 (2018).
3. N. Siemonsen, J. Steinhoff and J. Vines, *Phys. Rev. D* **97**, 124046 (2018).
4. M. Levi and J. Steinhoff, *JHEP* **09**, 219 (2015).
5. L. Blanchet, *Living Rev. Relativity* **17**, 2 (2014).
6. Y. Pan, A. Buonanno, R. Fujita, E. Racine and H. Tagoshi *Phys. Rev. D* **83**, 064003 (2011).
7. Y. Pan, A. Buonanno, A. Taracchini, L. E. Kidder, A. H. Mroué, H. P. Pfeiffer, M. A. Scheel and B. Szilágyi, *Phys. Rev. D* **89**, 084006 (2014).