A new class of non-aligned Einstein-Maxwell solutions with a geodesic, shearfree and non-expanding multiple Debever-Penrose vector

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In a recent study of algebraically special Einstein-Maxwell fields [1] it was shown that, for non-zero cosmological constant, non-aligned solutions cannot have a geodesic and shearfree multiple Debever-Penrose vector $\boldsymbol{k}$. When $\Lambda = 0$ such solutions do exist and can be classified, after fixing the null-tetrad such that $\Psi_0 = \Psi_1 = \Phi_1 = 0$ and $\Phi_0 = 1$, according to whether the Newman-Penrose coefficient π is 0 or not. The family $\pi = 0$ contains the Griffiths solutions [2], with as sub-families the Cahen-Spelkens, Cahen-Leroy and Szekeres metrics. It was claimed in Ref. 2 (and repeated in Ref. 1) that for $\pi = 0$ both null-rays $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are necessarily non-twisting ($\bar{\rho} - \rho = \bar{\mu} - \mu = 0$): while it is certainly true that $\mu(\bar{\rho} - \rho) = 0$, the case $\mu = 0$ appears to have been overlooked. I reduce the sub-family in which $\boldsymbol{k}$ is non-expanding ($\rho + \bar{\rho} = 0$) to an integrable system of pde's and I present an explicit family of solutions.

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1. Introduction

In the quest for exact solutions of the Einstein-Maxwell equations

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \tag{1}$$

a large amount of research has been devoted to the study of so called *aligned* Einstein-Maxwell fields, in which at least one of the principal null directions (PND's) of the electromagnetic field tensor $\boldsymbol{F}$ is parallel to a PND of the Weyl tensor, a so called Debever-Penrose (DP) direction. One of the main properties in this respect is the Goldberg-Sachs theorem [3], stating that, if a space-time admits a complex null tetrad $(\boldsymbol{k}, \boldsymbol{\ell}, \boldsymbol{m}, \overline{\boldsymbol{m}})$ such that $\boldsymbol{k}$ is shear-free and geodesic and $R_{ab}k^a k^b = R_{ab}k^a m^b = R_{ab}m^a m^b = 0$ (as is the case when $\boldsymbol{k}$ is also a PND of $\boldsymbol{F}$), then the Weyl tensor is algebraically special, with $\boldsymbol{k}$ being a multiple Weyl-PND. One of the topics considered in Ref. 1, dealing with the reverse problem, enquired after the existence of algebraically special (non-conformally flat and non-null) Einstein-Maxwell fields with a possible non-zero cosmological constant for which the multiple Weyl-PND $\boldsymbol{k}$ is geodesic and shear-free ($\Psi_0 = \Psi_1 = \kappa = \sigma = 0$) and for which $\boldsymbol{k}$ is *not* parallel to a PND of $\boldsymbol{F}$ ($\Phi_0 \neq 0$). Choosing a null-rotation about $\boldsymbol{k}$ such that $\Phi_1 = 0$, it follows that $\Phi_2 \neq 0$: with $\Phi_2 = 0$ $\boldsymbol{\ell}$ would be geodesic and shear-free and the Goldberg-Sachs theorem would imply $\Psi_3 = \Psi_4 = 0$. The Petrov type would then be D, in which case [4,5] the only null Einstein-Maxwell solutions are given by the (doubly aligned) Robinson-Trautman metrics.

Using the GHP formalism[a] the Maxwell equations and Bianchi equations become then

$$\begin{aligned}
ð'\Phi_0 &= -\pi\Phi_0, &(2)\\
þ\Phi_0 &= 0, &(3)\\
þ'\Phi_0 &= -\mu\Phi_0, &(4)\\
ð\Phi_2 &= -\nu\Phi_0 + \tau\Phi_2, &(5)\\
þ\Phi_2 &= -\lambda\Phi_0 + \rho\Phi_2, &(6)\\
ð\Psi_2 &= -\pi\Phi_0\overline{\Phi_2} + 3\tau\Psi_2, &(7)\\
þ\Psi_2 &= \mu|\Phi_0|^2 + 3\rho\Psi_2, &(8)
\end{aligned}$$

after which the commutators $[ð', ð], [ð', þ], [ð, þ'], [ð', þ']$ and $[þ', þ]$ applied to Φ_0 give

$$\begin{aligned}
ð\pi &= (3\rho - \overline{\rho})\mu - 2\Psi_2 + \frac{R}{12}, &(9)\\
þ\pi &= 3\rho\pi, &(10)\\
ð\mu &= \overline{\lambda}\pi + 3\mu\tau, &(11)\\
þ'\pi - ð'\mu &= 2\nu\rho - 2\lambda\tau - \pi\overline{\mu} - \mu\overline{\tau} - 2\Psi_3, &(12)\\
þ\mu &= \pi(\overline{\pi} + 3\tau) + 2\Psi_2 - \frac{R}{12}. &(13)
\end{aligned}$$

Herewith one of the GHP equations becomes a simple algebraic equation for Ψ_2,

$$\Psi_2 = \rho\mu - \tau\pi + \frac{R}{12}, \qquad (14)$$

the þ derivative of which results in $\rho R = 0$.

As $\rho = 0$ would imply $\Phi_0 = 0$, this leads to the remarkable consequence that an algebraically special Einstein-Maxwell solution possessing a shear-free and geodesic multiple Weyl-PND which is not a PND of $\boldsymbol{F}$ necessarily has a vanishing cosmological constant. The corresponding class of solutions is non-empty, as it contains the Griffiths[2] metrics, encompassing as special cases the metrics of Ref. 7, 8, 9, 10.

In Ref. 2 it was claimed that for $\pi = 0$ both null-rays $\boldsymbol{k}$ and $\boldsymbol{\ell}$ are necessarily non-twisting ($\bar{\rho} - \rho = \bar{\mu} - \mu = 0$). As a consequence it was also claimed in Ref. 1 that the Griffiths metrics are uniquely characterised by the condition $\pi = 0$. However, when $\pi = 0$ the only conclusion to be drawn from (10, 14) is that $\mu(\bar{\rho} - \rho) = 0$. When ρ is real this indeed leads to the metrics of Ref. 2, while the case $\mu = 0$ appears to have been overlooked and leads, as shown in the section below, to new classes of solutions.

[a]Throughout I use the sign conventions and notations of Ref. 6 §7.4, with the tetrad basis vectors taken as $\boldsymbol{k}, \boldsymbol{\ell}, \boldsymbol{m}, \overline{\boldsymbol{m}}$ with $-k^a\ell_a = 1 = m^a\overline{m}_a$. In order to ease comparison with the (more familiar) Newman-Penrose formalism, I will write primed variables, such as κ', σ', ρ' and τ', as their NP equivalents $-\nu, -\lambda, -\mu$ and $-\pi$.

2. The missing class

When $\pi = 0$ and $\mu = 0$ the equations from the previous paragraph immmediately lead to $\Psi_0 = \Psi_1 = \Psi_2 = 0$ and $\Psi_3 = \rho\nu - \lambda\tau$. We try to make some progress by looking for solutions for which $\boldsymbol{k}$ is non-expanding ($\rho + \overline{\rho} = 0$). Acting on this condition with the ð and Þ operators, the GHP equations lead to $\tau = 0$ and

$$\rho^2 + |\Phi_0|^2 = 0, \tag{15}$$

the ð derivative of which implies $\lambda = \Phi_2\overline{\Phi_0}\rho^{-1}$. We translate these results into Newman-Penrose (NP) language and fix a boost and spatial rotation in the $\boldsymbol{k},\boldsymbol{\ell}$ and $\boldsymbol{m},\overline{\boldsymbol{m}}$ planes such that $\Phi_0 = 1$ and $\rho = i$. It follows that the only non-0 spin coefficients are ρ, ν and $\lambda = -i\Phi_2$, with the only non-vanishing components of the Weyl spinor being $\Psi_3 = i\nu$ and Ψ_4. As $[D,\Delta] = 0$ coordinates u, v and $\zeta, \overline{\zeta}$ exist such that $D = \partial_u$, $\Delta = \partial_v$ and

$$\delta = e^{-iu}(\xi\partial_\zeta + \eta\partial_{\overline{\zeta}} + P\partial_u + Q\partial_v),$$

ξ,η,P,Q being arbitrary functions. The e^{-iu} factor is included for convenience: applying the $[\delta, D]$ commutator to u, v and ζ shows that ξ,η,P,Q are functions of $v,\zeta,\overline{\zeta}$ only. Introducing new variables $n = e^{-iu}\nu$ and $\phi = e^{-2iu}\Phi_2$ it follows that also n and ϕ depend on $v,\zeta,\overline{\zeta}$ only, with the full set of Jacobi equations and field equations reducing to the following system of pde's:

$$P_v + i\overline{P\phi} - \overline{n} = 0, \tag{16}$$
$$Q_v + i\overline{Q\phi} = 0, \tag{17}$$
$$\xi_v + i\overline{\eta}\overline{\phi} = 0, \tag{18}$$
$$\eta_v + i\overline{\xi}\overline{\phi} = 0, \tag{19}$$

$$e^{-iu}\overline{\delta}P - e^{iu}\delta\overline{P} - 2i|P|^2 = 0, \tag{20}$$
$$e^{-iu}\overline{\delta}Q - e^{iu}\delta\overline{Q} - 2i(\Re Q\overline{P} - 1) = 0, \tag{21}$$
$$e^{-iu}\overline{\delta}\xi - e^{iu}\delta\overline{\eta} - i(\xi\overline{P} + \overline{\eta}P) = 0, \tag{22}$$
$$\tag{23}$$

$$e^{iu}\delta n = -iPn + 2|\phi|^2, \tag{24}$$
$$e^{iu}\delta\phi = -2iP\phi - n, \tag{25}$$
$$\tag{26}$$

with the Ψ_4 component of the Weyl spinor given by $\Psi_4 = ie^{2iu}(\overline{n}\overline{P} + \Delta\phi) + e^{iu}\overline{\delta}n$. This system is integrable and a simple solution is obtained by assuming $\phi = H^2$ to be a positive constant: in terms of new coordinates u,v,a,b the null tetrad is given by

$$\boldsymbol{\omega}^1 = \frac{e^{iu}}{2H(a+b)}[e^{i\frac{\pi}{4}-v}\mathrm{d}a - e^{-i\frac{\pi}{4}+v}\mathrm{d}b], \tag{27}$$

$$\boldsymbol{\omega}^3 = \frac{1}{H^2}[\mathrm{d}v - \frac{1}{2(a+b)}\mathrm{d}(a-b)], \tag{28}$$

$$\boldsymbol{\omega}^4 = \mathrm{d}u - \frac{1}{2(a+b)}[e^{-2v}\mathrm{d}a - e^{2v}\mathrm{d}b]. \tag{29}$$

The corresponding space-time metric is

$$\mathrm{d}s^2 = \frac{1}{H^2}[-2\mathrm{d}v + \frac{1}{a+b}\mathrm{d}(a-b)]\mathrm{d}u + \frac{1}{H^2(a+b)}(e^{-2v}\mathrm{d}a - e^{2v}\mathrm{d}b)\mathrm{d}v + \frac{\cosh 2v}{H^2(a+b)^2}\mathrm{d}a\mathrm{d}b. \tag{30}$$

and the Maxwell field and energy-momentum tensor are obtained as

$$\boldsymbol{F} = iH^2(\boldsymbol{\omega}^1 - \boldsymbol{\omega}^2)\wedge\boldsymbol{\omega}^3 + i(e^{-2iu}\boldsymbol{\omega}^1 - e^{2iu}\boldsymbol{\omega}^2)\boldsymbol{\omega}^4, \tag{31}$$

$$\boldsymbol{T} = 2H^2(e^{-2iu}{\boldsymbol{\omega}^1}^2 + e^{2iu}{\boldsymbol{\omega}^2}^2 + H^2{\boldsymbol{\omega}^3}^2) + 2{\boldsymbol{\omega}^4}^2. \tag{32}$$

The Petrov type is III and the space-time admits three Killing vectors, $\partial_u, \partial_a - \partial_b$ and $a\partial_a + b\partial b$. At first sight it seems odd that the Weyl spinor components Ψ_3 and Ψ_4 depend on u, while the frame has been "invariantly" fixed. The explanation however is that the the frame was fixed by means of a null rotation putting $\Phi_0 = 1$, while the Maxwell field itself does not share the space-time symmetries: $\boldsymbol{F}$ is not Lie-propagated along the integral curves of the null Killing vector ∂_u (this also shows that the family of solutions presented here is distinct from the Einstein-Maxwell solutions admitting null-Killing vectors of Ref. 11.

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All calculations were done using the Maple symbolic algebra system, while the properties of the metric (30) were checked with the aid of Maple's DifferentialGeometry package .

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