# A new class of LRS spacetimes: General solutions and properties

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The spacetimes that are *Locally Rotationally Symmetric* (LRS) have been studied in detail and discussed many times in the literature in the cosmological context, i.e. with a fluid matter source (see for example [1, 2, 4] and the references therein). For these spacetimes there exists a continuous isotropy group at each point and hence there is a multiply-transitive isometry group acting on the spacetime manifold. As we know, the isotropies around a point in a spacetime with a fluid can occur as a one-dimensional or three dimensional subgroup of the full group of isometries that leaves the normalised 4-velocity of the matter flow invariant. A three dimensional group of isotropies at each point implies that the spacetime is isotropic at every point and gives rise to the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) models. While a one dimensional group of isotropies at each point corresponds to anisotropic and in general spatially inhomogeneous models[1], but includes also some spatially homogeneous (Bianchi and Kantowski-Sachs) models [5, 6]. The important property of LRS spacetimes is that they exhibit locally (at each point) a unique preferred spatial direction, covariantly defined, (for example, by a vorticity vector field, a non-vanishing non-gravitational acceleration of the matter, or a density gradient).

LRS spacetimes with a perfect fluid matter source have been completely analysed and classified by Stewart and Ellis using tetrad methods [2]. Using a semi-tetrad covariant formalism it was shown that the Einstein field equations can be written as a set of first order equations of geometrical scalars [4, 9]. By analysing the consistency conditions of the field equations, it was rigorously proven that a perfect fluid LRS spacetime cannot have a simultaneous fluid rotation and spatial twist of the preferred spatial direction. Based on this observation, the perfect fluid LRS spacetimes can be divided into three distinct classes. *Class I* spacetimes are those where the rotation is non zero but the twist vanishes. This class was shown to be non-expanding, non-distorting and stationary and the solutions generalise the well known Gödel solution. In *Class II* spacetimes both the rotation and the twist vanish and these consist of the spherical, hyper-spherical, and plane symmetric (cylindrical) solutions. *Class III* spacetimes have no rotation or acceleration but non-zero twist of the preferred spatial direction. These spacetimes are spatially homogeneous. However for an imperfect fluid - for example if there is an entropy flux - both can be non-zero.

By relaxing the condition for a perfect fluid, we transparently showed that it is possible to have a Locally Rotationally Symmetric spacetime with non-zero rotation and spatial twist simultaneously if we allow for non-zero and bounded heat flux. We investigated in detail all the covariant geometrical properties of such spacetimes and

[1] generically with one or two centres where the isotropy group is 3-dimensional

proved an interesting result that evolution of all the covariant scalars obey a single common hyperbolic linear second order partial differential equation. The existence of spacelike Cauchy surface, where initial Cauchy data can be provided is guaranteed. It was also shown that these solutions are self-similar as they possess a conformal Killing vector in the $[u, e]$ plane

As these solutions are neither stationary nor spatially homogeneous in general, with suitable equations of state, perhaps with the temperature $T$ as an internal variable in the equations of state for $P$, $\Pi$, and $Q$, they have the potential to give exact general relativistic models for rotating and dynamic and radiating stellar structures as they definitely have non zero heat flux in the interior. The radiative heat flux can be prominently seen for neutron stars, as for a newly formed neutron star the core temperature is of the order of $10^{11}K - 10^{12}K$, that rapidly drops to $10^6 K$ within a few years [12, 13]. This huge amount of heat transfer from the core to the surface of the neutron stars is due to several processes, like transmission of neutrino's, electron sound wave coupled with electromagnetic radiations in the superfluid stellar core [14]. Even for the main sequence stars (like sun), the radiative heat transfer from core to the convection zone is always present, albeit with an extremely high opacity [15]. Therefore we see that while studying the interior of an realistic astrophysical stars, heat flux does play a very important role and assuming a perfect fluid form of matter in these cases may lead to oversimplification. Hence any solution to the Einstein filed equations, that incorporate rotation, spatial twist and heat flux simultaneously (such as those reported in this paper), is definitely a better candidate to provide a relativistic description of a rotating stellar interior with quadrupole and other higher multipole moments. Also these may account for physical features of stars that cannot be explained by Newtonian dynamics.

By using the property of the self-similarity, we obtained a general solution to the field equations for the LRS spacetimes with non-zero rotation and spatial twist. We showed that we may specify an equation of state for the isotropic pressure at an initial Cauchy surface for particular applications.

This class of solutions have some very interesting properties, for both cosmological and stellar collapse scenarios. We can immediately see that there is a spacetime singularity along the curve $B\rho + A\tau = 0$, which is similar to the cosmological singularity of the FLRW or Lemaitre-Tolman-Bondi universes (or corresponding black hole singularities if we take the collapsing branch of the solutions). Apart from this, there are no other singular points on the manifold.

1. The most interesting feature of the singularity in this class of spacetime is it can be made timelike, spacelike or null by choice of the ratio of the constants $A$ and $B$. In other words, the ratio of rotation ($\Omega$) and spatial twist ($\xi$) at any initial Cauchy surface completely determines the nature of the initial (or final) singularity and this gives a range of different possibilities

2. For the cosmological scenario, let us consider both $A$ and $B$ to be greater than zero, In that case the initial singularity is along the line $B\rho + A\tau = 0$. This '*Big Bang*' is no longer instantaneous, and can be spacelike, timelike or null. Thus the section of the manifold that depicts the universe is given by

$$\rho > 0 \ , \ \ \tau > -(B/A)\rho \ . \tag{1}$$

For an expanding universe with positive energy density, we must have $\Theta > 0$ and $\mu > 0$, and hence we must choose the constants

$$C_\Theta > 0 \;\; ; \;\; C_\mu > 0 \; . \tag{2}$$

For the cosmological case we can choose *dustlike* matter with

$$p_0 = 0 \; , \tag{3}$$

$$C_\Pi = 0 \Rightarrow \Pi_0 = 0. \tag{4}$$

There is no bounce in this cosmology as the expansion goes to zero asymptotically. Furthermore it is interesting to note that at spacelike infinity '$i_0$' (where $\rho \to \infty$), timelike infinity '$i+$' (where $\tau \to \infty$) and future null infinity $\mathcal{I}+$, all the kinematical and dynamical quantities vanish, making the spacetime asymptotically Minkowski. Hence, we get a cosmology that is *Future asymptotically simple.*

3. Another interesting case happens when the curves $B\rho + A\tau =$ const. are null. In this case the initial singularity is *incoming null.* Then for any observer on the worldline $\rho = 0, (\tau > 0)$, observation along the past null cone will depict a universe with homogeneous density, in contrast to the fact that on a given time slice the density is inhomogeneous.

4. A similar picture can be obtained for collapsing stellar configurations with $A < 0$ and $B > 0$. In that case the section of the manifold $\rho > 0$ and $\tau < (B/|A|)\rho$ depicts the regular collapsing region which is *Past asymptotically simple.* To get a collapsing branch of the solution with positive matter density we must have $\Theta < 0$ and $\mu > 0$. Hence we choose

$$C_\Theta < 0 \;\; ; \;\; C_\mu > 0 \; . \tag{5}$$

Also here we should specify the equation of state linking the isotropic pressure to other thermodynamic variables and separately specify the constant $C_\Pi$ at the initial Cauchy surface subject to the energy conditions. We can easily check that in this case $\dot{\Theta} < 0, \dot{\mu} > 0$. Hence the collapse continues till $\Theta \to -\infty$ and $\mu \to \infty$. This is a final singularity at $\tau = (B/|A|)\rho$ and we can easily see that this singularity can be timelike, spacelike, or null, which will have important consequences in terms of the cosmic censorship conjecture.

We proved that a locally rotationally symmetric spacetime with simultaneous rotation and spatial twist is always self-similar as it has a homothetic Killing vector in the plane spanned by the fluid flow lines and the preferred spatial direction. This homothetic Killing vector becomes a null Killing vector in the special case when the non-zero and bounded heat flux reaches it's extremal value. Also when either the rotation or the spatial twist vanish, this homothetic vector becomes a timelike or spacelike killing vector making the spacetime either stationary or spatially homogeneous. However when both these quantity vanish there is no inherent Killing or conformal symmetry in the plane spanned by the fluid flow lines and the preferred spatial direction.

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