Equal-field cylindrical electrovacuum universes

V. Balek

Department of Theoretical Physics, Comenius University,
Mlynská dolina, 842 48 Bratislava, Slovakia
E-mail: balek@fmph.uniba.sk

M. Čermák

Institute of Physical Electronics, Masaryk University,
Kotlářská 267/2, 611 37 Brno, Czech Republic
E-mail: 63855@seznam.cz

We explore a family of solutions to Einstein-Maxwell equations with cylindrical symmetry, in which both metric and electromagnetic field are expressed in terms elementary functions. This is achieved by choosing the integration constants in (00) and (22) components of Einstein equations equal to each other, which is equivalent to putting electric field (transversal) and magnetic field (longitudinal), under suitable rescaling of coordinates, equal to each other. We discuss the connection between our family of solutions and previously known solutions summarized in the 1983 paper by MacCallum, analyze properties of spacetime described by our solutions and show how the equal-field condition can be relaxed perturbatively.

Keywords: Einstein-Maxwell equations; cylindrical symmetry; perturbation theory.

1. Einstein-Maxwell equations

Consider cylindrical version of Weyl-Lewis-Papapetrou metric,

$$ds^2 = \frac{1}{F}(dt + Ad\phi)^2 - F[B(dr^2 + dz^2) + w^2 d\phi^2], \tag{1}$$

with the electromagnetic vector potential of the form

$$\boldsymbol{A} = A_0 \boldsymbol{dt} + A_2 \boldsymbol{d\phi}, \tag{2}$$

where the functions F, A, B and w as well as A_0 and A_2 depend only on the coordinate r [1].

Due to high symmetry of the problem, we can perform one integration of all Maxwell equations and all Einstein equations but one, reducing them to differential equations of first order. For Maxwell equations it is seen immediately and for Einstein equations we can use identity

$$\xi^{\mu\nu}{}_{;\nu} = 2R^{\mu}_{\nu}\xi^{\nu}, \tag{3}$$

where $\xi_{\mu\nu}$ is a bivector associated with Killing vector ξ^{μ}, $\xi_{\mu\nu} = \xi_{\nu,\mu} - \xi_{\mu,\nu} = -2\xi_{\mu;\nu}$, and $R_{\mu\nu}$ is Ricci tensor. The integrated equations were obtained first with the help of Hamiltonian formalism[2], and then with the help of complex potentials[3] introduced in the series of papers[4–6]. Neither procedure used the identity (3) explicitly, although in the paper[6] the authors mentioned it with a reference to Synge's book[7].

First integral of Maxwell equations is

$$\sqrt{-{}^2g}A'_A = g_{AB}C^B, \tag{4}$$

and first integral of Einstein equations (five of six) is

$$\sqrt{-{}^2g}g^{ac}g'_{cb} = 2\delta^a_A\delta^B_b C^A A_B - \delta^a_b C^C A_C + K_b{}^a, \tag{5}$$

where the indices $A, B, \ldots$ assume the values 0 and 2, the indices $a, b, \ldots$ assume the values 0, 2, 3 (the coordinates are identified in a usual way as $(x^0, x^1, x^2, x^3) = (t, r, \phi, z)$), ${}^2g$ is the determinant of the (t, ϕ) part of the metric, ${}^2g = \det(g_{AB}) = -w^2$, and C^A, $K_b{}^a$ are constants. After the $\binom{1}{1}$ component of Einstein equations is added to the system, we have eight equations for six functions F, A, B, w, A_0, A_2; thus, two equations are just constraints on integration constants.

The trace of the $\binom{A}{B}$ part of (5) yields $w = (1/2)K_A{}^A r$, so that by rescaling r (as well as z, in order to preserve the quadratic form $dr^2 + dz^2$) we obtain $w = r$. If we also gauge away the constants $K_0{}^2$ and $K_2{}^0$, in the $\binom{A}{B}$ part of (5) there remains a single constant $C = (1/2)(K_0{}^0 - K_2{}^2)$. The constraint on C can be turned into an independent equation by inserting expressions for A_A in terms of A, A', F, F' into it, and equations (4) can be then reinterpreted as constraints. However, one of these equations is satisfied identically, so that we are left with one constraint only.

2. Equal-field solution

We are interested in solutions to equations (4), (5) with $C = 0$. Their subclass was found long ago by using transformation to rotating frame in which the metric was static[8]. Later it was noted[3,9] that the subclass belongs to a broader family with constant ratio $\mathcal{R} = A'_2/A'_0$. It has $\mathcal{R} = -1/k$, where $k = C^0/C^2$, while a class of solutions proposed in[10] has another distinguished value $\mathcal{R} = -k$. We construct solutions with arbitrary

$\mathcal{R}$ by solving the equations directly; however, our solutions can be obtained also from solutions with $\mathcal{R} = -1/k$ by means of the scaling discussed below.

Introduce a new function $f = (Fr/A_+)^{1/2}$, where $A_+ = A + k$. Einstein equations yield a pair of coupled equations for the functions f and A,

$$\frac{f''}{f} - \frac{p}{2A_+ r^2}\Big(1 + \frac{3}{2}\frac{rA'}{A_+}\Big) - \frac{q + 1/2}{2r^2} = 0, \quad r[A_+(f^4 - 1)]' + pf^4 = 0, \quad (6)$$

where $p = 2kC$ and $q = K_3^3 - C$. For $C = 0$ we have $p = 0$, so that the first equation (6) decouples from the second equation and can be solved analytically. The solution is

$$f = r^{1/2} \times \begin{cases} (k_1 r^a + k_2 r^{-a}), \text{ if } a > 0 \\ (k_1 \ln r + k_2), \text{ if } a = 0 \\ [k_1 \cos(\hat{a} \ln r) + k_2 \sin(\hat{a} \ln r)], \text{ if } a = i\hat{a} \end{cases}, \quad (7)$$

where $a = \sqrt{(1 + q)/2}$. The function A_+ is obtained immediately from the second equation (6), the function F appears in the definition of f, for the function $\mathcal{B} = BF$ we have separate equation and the functions A_A are given algebraically in terms of A, A', F, F'. In this way we find

$$A_+ = \frac{k_A}{f^4 - 1}, \quad Fr = \frac{k_A f^2}{f^4 - 1}, \quad \mathcal{B} = k_B r^q f^2, \quad (8a)$$

$$A_0 = \frac{2}{C^2 k_A}\frac{rf'}{f}, \quad A_2 = -(k_A + k)A_0. \quad (8b)$$

Finally, upon inserting the expressions for A_0, F and A into the first equation (4), we find

$$\begin{cases} 4a^2 k_1 k_2 \\ -k_1^2 \\ -\hat{a}^2(k_1^2 + k_2^2) \end{cases} = \frac{1}{2}(C^2)^2 k_A, \text{ if } \begin{cases} a > 0 \\ a = 0 \\ a = i\hat{a} \end{cases}. \quad (9)$$

The solution preserves its form under the scaling $t \to \gamma t$ and $(r, z) \to \gamma^{-1}(r, z)$, if f stays unchanged, which can be always achieved by suitably scaling (k_1, k_2), and if the remaining constants scale as $(k, k_A) \to \gamma(k, k_A)$ and $k_B \to \gamma^{q+2} k_B$. In case the constant $\kappa = k_A + k$ is nonzero, we can use the procedure to normalize it to ± 1 by fixing the unit of length and choosing $\gamma = |\kappa|^{-1}$, and as a result, to make the potentials A_0 and A_2 as well as the field strengths F_{01} and F_{21} to be equal in absolute value. In that sense we call the solution (8) "equal-field solution". Note, however, that the field strengths are the same only when computed with respect to *coordinate* basis. The physical field strengths which refer to the orthonormal basis $e_{\hat{\alpha}}{}^{\mu}$ carried by the observer at rest, $E^{\hat{1}} = F_{\hat{0}\hat{1}} = \sqrt{F/\mathcal{B}} F_{01}$ and $B^{\hat{3}} = F_{\hat{2}\hat{1}} = (1/\sqrt{F\mathcal{B}})(F_{21} - AF_{01})/r$, differ from each other.

3. Properties of spacetime

The (t, ϕ) part of the metric can be expressed in two different ways,

$$ds_2^2 = \frac{1}{F}(dt + Ad\phi)^2 - Fr^2 d\phi^2 = \frac{1}{\mathcal{F}} dt^2 - \mathcal{F} r^2 (d\phi - \omega dt)^2, \qquad (10)$$

where the functions $\mathcal{F}$, ω are defined in terms of the function $\mathcal{A} = A^2 - F^2 r^2$ as $\mathcal{F} = -\mathcal{A}/(Fr^2)$, $\omega = -A/\mathcal{A}$. The cylinders with $1/F = 0$ are *static limits* (surfaces separating regions in which observers can stay at rest from regions in which they must rotate) and the cylinders with $\mathcal{F} = 0$ are *chronological limits* (surfaces separating regions in which time travel is forbidden from regions in which it is allowed). Suppose $a > 0$, so that $f = (r/r_1)^{a_+} \pm (r/r_2)^{-a_-}$, where $a_\pm = a \pm 1/2$ and the sign in front of the second term is given by the sign of k_A. For $k_A > 0$, $a > 1/2$ as well as for $k_A < 0$, $a < 1/2$ the function f has one minimum, defining a cylinder composed of photon orbits; in the latter case it has also one zero, defining a singular "cylinder" (in fact, a line parallel to the z axis).

The behavior of light cones in equal-field solution is depicted in fig 1. In

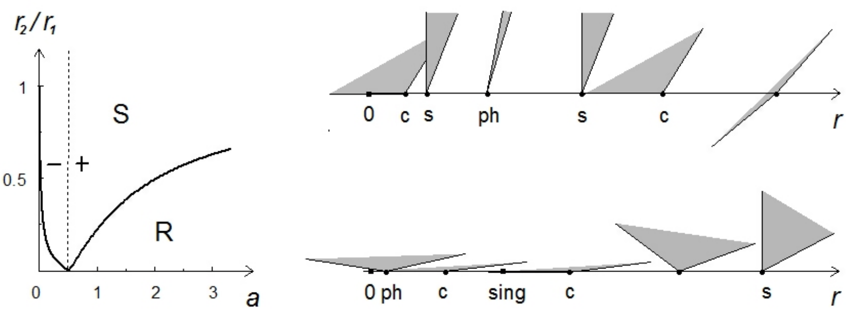

Fig. 1. Regions in parametric space with and without rotating domain in the neighborhood of photon orbits (left) and light cones in two typical spacetimes (right).

the left panel, regions in the plane $(a, r_2/r_1)$ with static (S) and rotating (R) neighborhood of photon orbits are shown; in the right panel, light cones in metric with $k_A > 0$, $a > 1/2$ (upper diagram) and $k_A < 0$, $a < 1/2$ (lower diagram) are displayed at various values of r in the planes tangential to respective cylinders. The sign of k_A is denoted by $+$ and $-$ in the left panel and singular "cylinder", photon cylinder, static limits and chronology limits are denoted by *sing*, *ph*, *s* and *c* in the right panel.

4. Perturbing the metric

Equations for the functions f and $\mu = A_+^{-1}$, rewritten in the variable $\tau = \ln r$, read

$$\ddot{f} - \dot{f} - Qf = \frac{p}{2}\Big(\mu - \frac{3}{2}\dot{\mu}\Big)f, \quad \Big(\frac{f^4 - 1}{\mu}\Big)^{\cdot} = -pf^4, \tag{11}$$

where $Q = (1/2)(q + 1/2)$ and the dot denotes differentiation with respect to τ. Suppose the parameter p is small. In the zeroth order in p we have the previous expressions $f_0 = k_1 e^{a+\tau} + k_2 e^{-a-\tau}$ and $\mu_0 = k_A^{-1}(f_0^4 - 1)$, and if we write down the equations for the first corrections to f and A,

$$\ddot{f}_1 - \dot{f}_1 - Qf_1 = \frac{p}{2k_A}(f_0 - 6\dot{f}_0)f_0^4, \quad \Big[(f_0^4 - 1)A_1 + \frac{4f_0^3}{\mu_0}f_1\Big]^{\cdot} = -pf_0^4 \tag{12}$$

(we have skipped the term proportional to f_0 from the first equation, since it can be always removed by redefining Q), we find

$$f_1 = \frac{p}{2k_A}f_0\Phi_0, \quad A_1 = -p\frac{f_0^4 + 1}{(f_0^4 - 1)^2}\Phi_0, \tag{13}$$

where $\Phi_0 = \int f_0^4 d\tau$. To complete the calculation of the perturbed metric, we can determine F_1 from the definition of f and B_1 from the equation for $\mathcal{B}$. Finally, we can impose the first equation (4) as a constraint on the perturbed solution. With the notations $k_1 = k_{10}(1+p\lambda_1)$, $k_2 = k_{20}(1+p\lambda_2)$, $k_A = k_{A0}(1 + p\lambda_A)$ and $C^2 = C_0^2(1 + p\lambda_C)$, the resulting equation reads

$$\lambda_C = -\frac{1}{2}\Big(\lambda_1 + \lambda_2 - \lambda_A - \frac{1}{2a^2 k_{A0}}\Big). \tag{14}$$

References

1. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Hertl: *Exact Solutions of Einstein's Field Equations*, CUP, Cambridge (2003).
2. D. M. Chitre, R. Güven, Y. Nutku, J. Math. Phys. **16**, 475 (1975).
3. N. Van den Bergh, P. Wils, J. Phys. **A**, 3843 (1982).
4. W. Kinnersley, J. Math. Phys. **18**, 1529 (1977).
5. W. Kinnersley, D. M. Chitre, J. Math. Phys. **18**, 1538 (1977).
6. W. Kinnersley, D. M. Chitre, J. Math. Phys. **19**, 1926 (1978).
7. J. Synge: *Relativity: The General Theory*, North-Holland, Amsterdam (1960).
8. N. Arbex, M. M. Som, Il Nuovo Cim. **13 B**, 49 (1973).
9. M. MacCallum, J. Phys. **A16**, 3854 (1983).
10. J. D. McCrea, J. Phys. **A15**, 1587 (1982).