The circularly polarized beam of electromagnetic radiation in the Einstein-Maxwell theory

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We derive the metric of a beam of circularly polarized electromagnetic radiation from the Einstein-Maxwell equations. We show how the uniform plane wave solutions miss the angular momentum carried by the wave. We study the energy, momentum, angular momentum and their fluxes along the cylinder both for this beam and in general. The three Killing vectors of any stationary cylindrical system give three Komar flux vectors which in turn give six conserved fluxes. The contribution is based on the paper [1] published in December 2017 by J.B. and the late Donald Lynden-Bell.

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Introduction

The exact Einstein-Maxwell metric for the plane wave can be found in the literature see, for example, Ref. [2]. However there is a problem with uniform waves that extend to infinite distance from their propagation axis. The electromagnetic vectors $\mathbf{E}$, $\mathbf{B}$ are perpendicular both to each other and to the propagation vector $\mathbf{k}$ so the Poynting vector $\mathbf{\Pi} = \mathbf{E} \times \mathbf{B}/(4\pi c)$ lies along $\mathbf{k}$ so the angular momentum flux $\mathbf{R} \times \mathbf{\Pi}$ has no component along the direction of propagation. This is clearly wrong for circularly polarized waves. The apparent paradox is nicely resolved in Jackson's book [3]: the angular momentum in the direction of propagation lies at the edge of the beam where the intensity falls off.

To assess the angular momentum one must consider non-uniform beams of finite cross section. The metric of such a system was first considered by Bonnor in Refs. [4],[5] but with the light replaced by null dust. Our solution is of the general form found by Bonnor [5] for spinning null dust. We determine his free functions in terms of the electric field profile across the wave. For the beam of uniform intensity and finite cross section all the angular momentum is at the edge.

After giving the flat-space solution for a circularly polarized electromagnetic beam of finite cross-section, we discuss cylindrical boundary conditions for Einstein's equations. Then we give the general solution to the Einstein-Maxwell equations for such beams. In Ref.1 we give explicit solutions for beams with particular profiles. Therein we also analyze general questions concerned with detection of conserved quantities.

Monochromatic circularly polarized light

In flat space electromagnetic vectors satisfy

$$\nabla \cdot \mathbf{E} = 0,\ \nabla^2 \mathbf{E} = c^{-2}\partial^2 \mathbf{E}/\partial t^2,\ \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial ct. \tag{1}$$

If the wave travels in the z direction then $\mathbf{E} \propto \Re \exp[ik(z-ct)]$ and a solution of the equations (1) is

$$\begin{aligned} \mathbf{E} = \Re(\mathbf{E_c}), \quad \mathbf{B} = -\Re(i\mathbf{E_c}), \\ \mathbf{E_c} = E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\exp[ik(z-ct)], \end{aligned} \tag{2}$$

where E_0 is a constant. This electrodynamic field can also be described by the complex 4-vector potential $(A_c)_t = 0,\ \mathbf{A_c} = -\mathbf{E_c} i/k$. However such a wave fills all space and we wish to have a beam of finite cross section. So, following Jackson [3], we look for a solution that falls to zero at the edge with $E_0 = E_0(R),\ \ R^2 = x^2 + y^2$. To satisfy $\nabla \cdot \mathbf{E} = 0$ we take $\mathbf{E} = \Re\,[[E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + E_1\hat{\mathbf{z}}]\exp[ik(z-ct)]]$ and find

$$E_1 = (i/k)(\partial E_0/\partial x + i\partial E_0/\partial y) = i(kR)^{-1}(x+iy)E_0', \tag{3}$$

where $E_0' = dE_0/dR$. Thus our fields take the form

$$\begin{aligned} \mathbf{E} = \Re(\mathbf{E_c}), \quad \mathbf{B} = -\Re[i\mathbf{E_c}], \quad \mathbf{A_c} = -\mathbf{E_c} i/k, \\ \mathbf{E_c} = [E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + \frac{i\hat{\mathbf{z}}}{kR}E_0'(x+iy)]\exp[ik(z-ct)]. \end{aligned} \tag{4}$$

Because E_0 varies, (4) no longer satisfies (1) exactly, but provided $E_0(R)$ varies slowly so that it remains almost constant over the scale of one wavelength, then (4) remains an approximate solution with the amplitude varying slowly across the beam. The terms in equation (1) that we neglect are $(RE_0')'/R << k^2 E_0$ and the radial derivative of that inequality which is necessary for the $\hat{\mathbf{z}}$ terms. Thus provided E_0 varies only on the scale of the overall beam radius $R = a$ and $ka >> 1$ the errors will be of order $(ka)^{-2}$. Expressing the fields in real terms with $u = ct - z$,

$$\begin{aligned} \mathbf{E} = E_0[\ \hat{\mathbf{x}}\cos(ku) + \hat{\mathbf{y}}\sin(-ku)] \\ +(E_0'/k)\hat{\mathbf{z}}\sin[-ku+\varphi], \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbf{B} = E_0[-\hat{\mathbf{x}}\sin(ku) + \hat{\mathbf{y}}\cos(ku)] \\ +(E_0'/k)\hat{\mathbf{z}}\cos[ku-\varphi]. \end{aligned} \tag{6}$$

These fields make up the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The Poynting vector of the field (4) is

$$\mathbf{\Pi} = c\mathbf{E} \times \mathbf{B}/(4\pi) = \frac{c}{4\pi}\big[\frac{E_0 E_0'}{kR}(\mathbf{r} \times \hat{\mathbf{z}}) + E_0^2 \hat{\mathbf{z}}\big]; \tag{7}$$

the component around the beam has only one k in its denominator. We neglect terms of order $1/(ka)^2$ but keep terms of order $1/(ka)$. The z component of angular momentum density l_z is $\hat{\mathbf{z}} \cdot (\mathbf{r} \times \mathbf{\Pi}) = (\hat{\mathbf{z}} \times \mathbf{r}) \cdot \mathbf{\Pi}$, so

$$l_z = -\frac{E_0 E_0' R}{4\pi k c}. \tag{8}$$

For an almost uniform beam which falls off near the edge, E_0' is zero except near the edge where the angular momentum along the beam is thus concentrated.

The stress-energy tensor in cylindrical coordinates is (with $\alpha = E_0 E_0'/kR$)

$$4\pi \boldsymbol{T} = \begin{pmatrix} E_0^2 & 0 & \alpha R^2 & E_0^2 \\ \cdot & 0 & 0 & 0 \\ \cdot & \cdot & 0 & \alpha R^2 \\ \cdot & \cdot & \cdot & -E_0^2 \end{pmatrix}. \tag{9}$$

Boundary conditions on cylindrical metrics

We can define our boundary conditions for stationary cylindrical systems by the requirements that there exist coordinates (t, R, φ, z) (in which φ is an azimuth and R is the length of the corresponding Killing vector); further the metric components $g_{\varphi\varphi} = R^2$ by definition, $\xi^2 = g_{tt}$ should be $O[R^n]$ for some n, $g_{t\varphi}/\xi^2 = O[1]$, $g_{tz}/\xi^2 = O[1/R]$, $g_{RR} = O[R^n]$ for some n, $g_{\varphi z} = O[R]$, $g_{zz} = O[\ln R]$ all at large R, wherever the system extends to large R. We adopt these boundary conditions.

Relativistic metric of circularly polarized light

An empty rotating cylindrical shell produces a flat internal space but in axes that rotate relative to axes fixed at infinity. We conjecture that the gravitational internal solution will be the same as Bonnor's[4] internal solution (10) below. To agree with[2] we write Φ for Bonnor's A. For $R \le a$ we write (hereafter we set $c = 1$)

$$ds^2 = dt^2 - (dR^2 + R^2 d\varphi^2 + dz^2) + \Phi(dt - dz)^2, \tag{10}$$

but the external solution will show that this is now relative to rotating axes.

Bonnor's fine papers do not start from the Einstein-Maxwell equations. Instead[4] treats unpolarized light as null dust and in[5] generalises this to some sort of spinning null fluid. By contrast van Holten[2] starts from the exact Einstein-Maxwell equations but unlike Bonnor he is interested in infinite waves. Since the edge is missing, there is no angular momentum in these solutions; they need modification for a beam with a non-uniform amplitude $E_0(R)$. In[5] Bonnor gives a metric which has sufficient generality to solve our problem. However he does not give an interpretation of the physical meaning of his free functions or show how to specialise them to be the metric of a circularly polarized light beam that satisfies the Einstein-Maxwell equations. We do this here. Denoting the derivative of $\psi(R)$ by ψ', Bonnor's metric in our notation is

$$ds^2 = -dR^2 - R^2 d\varphi^2 + du d\mathrm{v} + \Phi du^2 - R\psi' d\varphi du, \tag{11}$$

where the functions Φ, ψ depend on R only. Comparing this to van Holten's exact Einstein-Maxwell wave, we conclude that ψ is zero for the infinite uniform wave, so in the slowly varying approximation ψ' is small and will be neglected when multiplied by E_0'. We also neglect E_0'' and $(E_0')^2$ as in the purely electromagnetic case. The metric (11) does not depend on v which is an affine parameter along the null rays (R, φ, u) =const. The Maxwell equations turn out to be satisfied in the curved space-time. The solution is exact for the uniform wave even when ψ is present in the metric. We now turn to the Einstein equations. We first find both $F^2 = 0 = FF^*$ and

$$4\pi T_{\mu\nu} = F_{\mu\sigma}F^{\sigma}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \cdot & 0 & RE_0E_0'/k & 0 \\ \cdot & \cdot & E_0^2 & 0 \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}. \tag{12}$$

Bonnor gives the $T_{\mu\nu}$ corresponding to his metric (11) but with x, y replacing R, φ and with the notational changes given earlier. Putting his result in our notation, we find his stress tensor has the form

$$\begin{aligned} \kappa T_{uu} &= \tfrac{1}{2}\left[\nabla^2\Phi + \tfrac{1}{4}(\nabla^2\psi)^2\right] = 2GE_0^2; \\ \kappa T_{\varphi u} = \kappa T_{u\varphi} &= -\tfrac{1}{2}\, R\partial_R \nabla^2\psi = 2GRE_0E_0'/k. \end{aligned} \tag{13}$$

This establishes that Bonnor's metric can be specialised to correspond to the gravity of a circularly polarized electromagnetic beam in the high frequency limit. Thus while the electrodynamics of a finite beam has to be done approximately, the ensuing gravitational calculation will give the exact metric of the approximate stress tensor. The approximation becomes exact in both the high frequency limit and for the uniform beam of infinite cross-section. We perform the integrations under the boundary condition that $R\psi'$ should not diverge at infinity. Integrating the Einstein equations we find

$$\psi = -2G \int_0^R \left[\frac{1}{kR}\int_0^R E_0^2 R dR\right] dR, \tag{14}$$

and

$$\Phi = G \int_0^R \left[\frac{1}{R}\int_0^R (4E_0^2 - GE_0^4/k^2)\; RdR\right] dR. \tag{15}$$

The formulae above give the metric for a circularly polarized beam of any profile. They satisfy the boundary conditions formulated above.

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