# Einstein anomaly for vector and axial-vector fields in six dimensional curved space

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By applying the covariant Taylor expansion method of the heat kernel, Einstein anomaly associated with the Weyl fermion of spin 1/2 interacting with nonabelian vector and axial-vector fields in six dimensional curved space are manifestly given. From the relation between Einstein and Lorentz anomalies, which are the gravitational anomalies, all terms of the Einstein anomaly should form total derivatives. It is shown before the trace operation of the gamma-matrices that the anomaly is expressed by the form expected.

Motivated by the quantum effects in supergravity, we study gravitational anomalies in higher dimensional curved space. In supergravity coupled with super Yang-Mills theory,[1,2] the Lagrangian contains four-fermion interactions, which are regarded as some two-fermion interactions with bosonic background fields expressed by odd-order tensors. The completely antisymmetric part of the highest order tensor should be rewritten as an axial-vector by contracting its tensor with the Levi-Civita symbol. The (polar-)vector and the axial-vector parts in the two-fermion interactions can be absorbed in the vector and the axial-vector gauge fields. The concrete form of the gravitational anomalies in the model may directly be calculated by using the heat kernel.[3]

The heat kernel $K^{\{d\}}(x, x')$ for a fermion of spin $\frac{1}{2}$ in $d$ dimensions defined by

$$\frac{\partial}{\partial t} K^{(d)}(x, x'; t) = -H K^{(d)}(x, x'; t), \tag{1}$$

$$K^{(d)}(x, x'; 0) = \mathbf{1} |h(x)|^{-\frac{1}{2}} |h(x')|^{-\frac{1}{2}} \delta^{(d)}(x, x'), \tag{2}$$

where $\delta^{(d)}(x, x')$ is the $d$-dimensional invariant $\delta$-function, $\mathbf{1} = \{\delta^A{}_B\}$ the unit matrix for the spinor, and $h = \det h^a{}_\mu$, in which $h^a{}_\mu$ is a vielbein. Here $H$ is the second order differential operator, corresponding to the square of the Dirac operator $\not{D}$ in the case of the fermion $\psi$,

$$H = \not{D}^2 = D_\mu D^\mu + X, \quad \not{D} = \gamma^\mu \nabla_\mu + Y, \quad D_\mu = \nabla_\mu + Q_\mu, \quad Q_\mu = \frac{1}{2}\{\gamma_\mu, Y\},$$
$$X = Z - \nabla_\mu Q^\mu - Q_\mu Q^\mu, \quad \nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}\, \omega^{ab}{}_\mu \, \gamma_{ab}\, \psi, \quad \gamma_{a_1 \cdots a_j} = \gamma_{[a_i} \cdots \gamma_{a_j]},$$
$$Z = \frac{1}{2}\, \gamma^{\mu\nu} \,[\nabla_\mu, \nabla_\nu] + \gamma^\mu \nabla_\mu Y + Y^2, \qquad [D_\mu, D_\nu]\psi = \Lambda_{\mu\nu}\, \psi, \tag{3}$$

where $\omega^{ab}{}_{\mu}$ is the Ricci's coefficient of rotation. When in $d = 2n$ dimensions the fermion interacts with vector and axial-vector fields which do not commute each other, the Dirac operator contains the coupling of these bosons in $Y$,

$$Y = \gamma^{\mu} V_{\mu} + \gamma_{2n+1}\gamma^{\mu} A_{\mu}, \quad V_{\mu} \equiv V^a_{\mu} T^a, \quad A_{\mu} \equiv A^a_{\mu} T^a, \quad \gamma_{2n+1} = i^n \gamma^1 \gamma^2 \cdots \gamma^{2n}. \tag{4}$$

Here the representation matrix $T^a$ of a gauge group, and $V^a_{\mu}$ ($A^a_{\mu}$) is pure imaginary (real), because of the hermiticity of the Dirac operator. The quantities $Q_{\mu}$, $X$ and $\Lambda_{\mu\nu}$ in (3) are expressed in the following tensorial form,

$$\begin{aligned}
Q_{\mu} &= V_{\mu} - \gamma_{2n+1}\,\gamma_{\mu\rho} A^{\rho}, \qquad F_{\mu\nu} = \partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu} + [V_{\mu}, V_{\nu}], \\
X &= -\frac{1}{4}\,R + 2(n-1)\,A_{\mu} A^{\mu} - \gamma_{2n+1} A^{\mu}{}_{;\mu} + \gamma^{\mu\nu}\left(\frac{1}{2}\,F_{\mu\nu} + \frac{2n-3}{2}\,[A_{\mu}, A_{\nu}]\right), \\
\Lambda_{\mu\nu} &= \frac{1}{4}\,\gamma^{\rho\sigma} R_{\rho\sigma\mu\nu} + F_{\mu\nu} - [A_{\mu}, A_{\nu}] - 2\,\gamma_{\mu\nu} A_{\rho} A^{\rho} + 2\,\gamma_{[\mu|}{}^{\rho}\{A_{|\nu]}, A_{\rho}\} \\
&\quad + 2\,\gamma_{2n+1}\,\gamma_{[\mu|\rho} A^{\rho}{}_{;|\nu]} - 2\,\gamma_{\mu\nu\rho\sigma} A^{\rho} A^{\sigma},
\end{aligned} \tag{5}$$

where $R_{\rho\sigma\mu\nu}$ denotes the curvature tensor, and the semi-colon ';' means the Riemannian covariant differentiation $\nabla_{\mu} + V_{\mu}$ with respect to the vector gauge field. The completely antisymmetric product $\gamma_{\mu\nu\rho\sigma}$ of $\gamma$-matrices in the last term of $\Lambda_{\mu\nu}$ is rewritten by $-\,\epsilon_{\mu\nu\rho\sigma}\gamma_5$ and $-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma\kappa\lambda}\gamma_7\gamma^{\kappa\lambda}$ in 4 and 6 dimensions, respectively.

The differential equation (1) of the heat kernel for the fermion interacting with the general boson fields is not solvable strictly. Therefore the heat kernel is usually calculated by using De Witt's ansatz [4], automatically satisfying (2),

$$K^{(2n)}(x, x'; t) \sim \frac{\Delta^{1/2}(x, x')}{(4\pi t)^n} \exp\left(\frac{\sigma(x, x')}{2t}\right) \sum_{q=0}^{\infty} a_q(x, x')\, t^q, \tag{6}$$

where $\sigma\,(x, x')$ is a half of square of the geodesic distance between $x$ and $x'$, $\Delta\,(x, x') = |h(x)|^{-1}|h(x')|^{-1}\det\{\nabla_{\mu}\nabla_{\nu'}\sigma\,(x, x')\}$, and $a_q(x, x')$ are bispinors. Note that the metric tensor in curved space is $g_{\mu\nu} = h^a{}_{\mu} h^b{}_{\nu}\eta_{ab}$ with $\eta_{ab} = -\,\delta_{ab}$ in flat tangent space, and that the coincidence limit of $a_0$ is $\lim_{x'\to x} a_0(x, x') \equiv [a_0](x) = \mathbf{1}$. The products of $\sigma_{;\mu}$ ($\equiv \nabla_{\mu}\sigma$) construct orthonormal bases $|n\rangle$ being the eigenfunctions for $\sigma^{;\nu} D_{\nu}$, and the bispinor $a_q$ can be expanded by the bases, [5]

$$\begin{aligned}
a_q &= \sum_{n=0}^{\infty} |n\rangle\langle n|a_q\rangle = \sum_{n} \frac{(-1)^n}{n!}\sigma^{;\mu'_1}\cdots\sigma^{;\mu'_n}\lim_{x\to x'}\left[D_{(\mu_1}\cdots D_{\mu_n)} a_q\right], \\
a_q\,(x, x') &= \langle 0|a_q\rangle\,(x') - \langle \mu|a_q\rangle\,(x')\,\sigma^{;\mu'}\,(x, x') + \cdots .
\end{aligned} \tag{7}$$

The gravitational anomalies are obtained in the case of a massless Weyl fermion $\psi_L$ in $2n$ dimensions. The formal expressions of two gravitational anomalies, *i.e.* the general coordinate anomaly $\mathcal{A}^{(2n)}_{\mu}$ and the Lorentz anomaly $\mathcal{A}^{(2n)}_{\mu\nu}$, are given from the path integral measure. [6] They are expressed by using the heat kernel

$K^{(2n)}(x,x';t)$ after the Gaussian cut-off regularization,

$$\begin{aligned} D^\mu \langle T_{\mu\nu}\rangle &= \mathcal{A}_\nu^{(2n)}, \qquad \langle T_{\mu\nu}\rangle_{\rm A} \equiv \frac{1}{2}(\langle T_{\mu\nu}\rangle - \langle T_{\nu\mu}\rangle) = \mathcal{A}_{\mu\nu}^{(2n)}, \\ \mathcal{A}_\nu^{(2n)}(x) &= -\frac{1}{2}\lim_{t\to 0}\lim_{x'\to x} \mathrm{Tr}\left\{\gamma_{2n+1}\left(D_\nu - D_{\nu'}\right)K^{(2n)}(x,x';t)\right\}, \\ \mathcal{A}_{\mu\nu}^{(2n)}(x) &= -\frac{1}{4}\lim_{t\to 0}\lim_{x'\to x} \mathrm{Tr}\left\{\gamma_{2n+1}\gamma_{\mu\nu}K^{(2n)}(x,x';t)\right\}, \end{aligned} \tag{8}$$

where Tr runs over both indices of $\gamma$-matrices and representation matrices of the gauge group. Since these anomalies simultaneously appear and are related to each other, $\mathcal{A}_\nu^{(2n)} = 2D^\mu \mathcal{A}_{\mu\nu}^{(2n)}$,[7] it seems that both general covariance and local Lorentz symmetry break down.

We consider the "pure" general coordinate anomaly $G_\mu$ is given by redefining the energy-momentum tensor density so that the local Lorentz symmetry is preserved,

$$D^\mu \langle T'_{\mu\nu}\rangle = D^\mu \langle T'_{\mu\nu}\rangle_S = G_\nu^{(2n)} = D^\mu \mathcal{A}_{\mu\nu}^{(2n)} = \frac{1}{2}\mathcal{A}_\nu^{(2n)}, \qquad \langle T'_{\mu\nu}\rangle_A = 0 \tag{9}$$

with $\langle T'_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle - \mathcal{A}_{\mu\nu}^{(2n)}$, where $\langle T'_{\mu\nu}\rangle_S$ is the symmetric part of the expectation varue of the energy-momentum tensor. The "pure" general coordinate anomaly in (9) is called as the Einstein anomaly. The "pure" Lorentz anomaly is also obtained by redefining the energy-momentum tensor density so that the general covariance is preserved,

$$\langle T''_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle - 2\mathcal{A}_{\mu\nu}^{(2n)}, \qquad D^\mu \langle T''_{\mu\nu}\rangle = 0, \qquad \langle T''_{\mu\nu}\rangle_A = -\mathcal{A}_{\mu\nu}^{(2n)}. \tag{10}$$

In order to perform the concrete calculation in $2n$ dimensions, the Einstein anomaly is rewritten by the expansion coefficients of $a_n$ in (7) and its derivatives,

$$G_\nu^{(2n)}(x) = -\frac{1}{4(4\pi)^n}\mathrm{Tr}\left\{\gamma_{2n+1}(2\langle\nu|a_n\rangle - \langle 0|a_n\rangle_{!\nu})(x)\right\}. \tag{11}$$

where the exclamation mark '!' means the modified covariant differentiation $D_\nu$. The anomaly in 4 dimensional curved space had already been derived,[8,9]

$$\begin{aligned} G_\nu^{(4)} &= -\frac{1}{64\pi^2}\mathrm{Tr}\big\{\gamma_5(2\langle\nu|a_2\rangle - \langle 0|a_2\rangle_{!\nu})\big\} = \frac{1}{192\pi^2}\mathrm{Tr}\,\gamma_5\left(\Lambda_{\mu\nu}X\right)^{!\mu} \\ &= \frac{1}{64\pi^2}\mathrm{tr}\Big[\epsilon_{\mu\nu\rho\sigma}\Big(\frac{1}{6}R^{\rho\sigma}{}_{\kappa\lambda}F^{\kappa\lambda} - \frac{1}{6}RF^{\rho\sigma} + \frac{1}{3}F^{\rho\sigma;\lambda}{}_\lambda \\ &\qquad + \frac{4}{3}\{A_\lambda, A^\rho\}F^{\lambda\sigma} + \frac{8}{3}A^\rho A^\sigma A_\lambda A^\lambda\Big) \\ &\qquad - \frac{4}{3}(F_{\mu\nu}A^\sigma{}_{;\sigma} + 2F_{[\mu|\lambda}A^\lambda{}_{;|\nu]}) + 8A_{[\mu}A_\nu A^\sigma{}_{;\sigma]}\Big]^{;\mu}, \end{aligned} \tag{12}$$

where "tr" means a trace over the representation matrices of the gauge group. A derivative term in $G_\nu^{(4)}$ before the trace operation of $\gamma$-matrices becomes some terms in tensorial form after the operation, and the Lorentz anomaly $\mathcal{A}_{\mu\nu}^{(4)}$ may easily be given from the resultant form of $G_\nu^{(4)}$ by the relation (9). Such properties of $G_\nu^{(4)}$

is succeeded in the case of $G_\nu^{(6)}$. Indeed, the straightforward calculation gives the concrete form of $G_\nu^{(6)}$ as expected,

$$
\begin{aligned}
G_\nu^{(6)} &= -\frac{1}{256\pi^3}\mathrm{Tr}\Big\{\gamma_7(2\langle\nu|a_3\rangle - \langle 0|a_3\rangle_{!\nu})\Big\} \\
&= -\frac{1}{256\pi^3}\mathrm{Tr}\Big\{\gamma_7\Big[\frac{1}{6}\Lambda_{\mu\nu}\left(\frac{1}{6}R+X\right)^2 + \frac{1}{45}J_{[\mu}X_{!\nu]} - \frac{1}{60}J_{[\mu!\nu]}X \\
&\qquad +\frac{1}{15}\Lambda_{\mu\nu}X_{!\rho}{}^{\rho} + \frac{2}{45}\Lambda_{\mu\nu!\rho}X^{!\rho} + \frac{1}{40}\Lambda_{\mu\nu!\rho}{}^{\rho}X + \frac{1}{180}[\Lambda_{\mu\rho},\Lambda_\nu{}^{\rho}]X \\
&\qquad +\frac{1}{180}R^{\rho}{}_{[\mu}\Lambda_{\nu]\rho}X + \frac{17}{360}R_{\mu\nu\rho\sigma}\Lambda^{\rho\sigma}X + \frac{1}{36}\Lambda_{\mu\nu}\Lambda_{\rho\sigma}\Lambda^{\rho\sigma} \\
&\qquad +\frac{1}{45}\Lambda_{[\mu}{}^{\rho}\Lambda_{\nu]}{}^{\sigma}\Lambda_{\rho\sigma} - \frac{1}{90}\Lambda_{\mu\nu!\rho}J^{\rho} + \frac{1}{45}\Lambda_{\rho[\mu}J_{\nu]}{}^{!\rho}\Big]^{;\mu}\Big\},
\end{aligned}
\tag{13}
$$

where $J^\rho \equiv \Lambda^{\sigma\rho}{}_{!\sigma}$. Some total derivative terms in $G_\nu^{(6)}$ before the trace operation will yield many terms in tensorial form, by using (5), and the derivation is still in progress.

If all $A_\mu$ are abelian in (12), then $G_\nu^{(4)}$ corresponds to the anomaly in space with torsion, which is originally expressed by the third order antisymmetric tensor. The dual vector of the tensor in four dimensions behaves as the axial-vector.[8] Note that the dual tensor of torsion in six or higher dimensions is the third or higher order antisymmetric tensor. In supergravity, there appear the contrtibutions of the vector, the axial-vector and the third order antisymmetric tensor fields, together, which do not commute. The anomaly with the vector and the axial-vector fields in six-dimensional space with nonabelian torsion may have the new terms containing the third order torsion tensor.

## References


1. S. Ferrara, F. Gliozzi, J. Scherk and P. van Nieuwenhuizen, Nucl. phys. **B117**, 333 (1976).
2. D. Z. Freedman and J. H. Schwarz, Phys. Rev. **D15**, 1007 (1977).
3. J. Schwinger, *Phys. Rev.* **82**, 664 (1951).
4. B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, 1965).
5. I. G. Avramidi, *Nucl. Phys.* **B355**, 712 (1991); S. Yajima, Y. Higasida, K. Kawano, and S.-I. Kubota, *Class. Quantum. Grav.* **16**, 1389 (1999).
6. K. Fujikawa, *Phys. Rev.* **D21**, 2848 (1980).
7. K. Fujikawa, M. Tomiya and O. Yasuda, *Z. Phys.* **C28**, 289 (1985).
8. S. Yajima and T. Kimura, Prog. Theor. Phys. **74**, 866 (1985).
9. S.-I. Kubota, S. Yajima, Y. Higasida and K. Kawano, *Class. Quantum. Grav.* **18**, 433 (2000).