Real-metric spacetime surfaces generated by 'hemix' spirals and validated by radar trajectories

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Pondering *Enrique Loedel*'s 1948 symmetrical dual spacetime chart whose angle's *sine* rather than *tan* reflects scaled velocity between inertial reference frames, led to a curious discovery in 2004: Angles of a spherical triangle *whose sines law ratio is one*, geometrise relativistic velocity composition. Unexplored in a related renowned 1909 paper by *Arnold Sommerfeld* and kept under wraps until publication of a recent book, the elementary germane criterion points to seemingly hitherto unknown hemispherical spirals which reflect the Gudermannian dependency of a fixed thrust rocket's home frame velocity on rocket clock time. This opens new paths for analysing relativistic acceleration contexts. 'Hemix'-generated *real-metric* $\lambda|\tau$ surfaces visualisable in $\mathbb{R}^3$ and vindicated by radar trajectory attributes—oddly a strategy rather unexploited in relativity—succinctly epitomise not only *Born's 'rigid motion' problem*, but also *non-Minkowski spacetime paradigms* such as *Bells' spaceships paradox* and other extended medium acceleration scenarios.

Keywords: relativistic spherical triangles; rigor mortis acceleration; real metric surface; Bell's spaceship problem; hemix spiral; hemicoid surface.

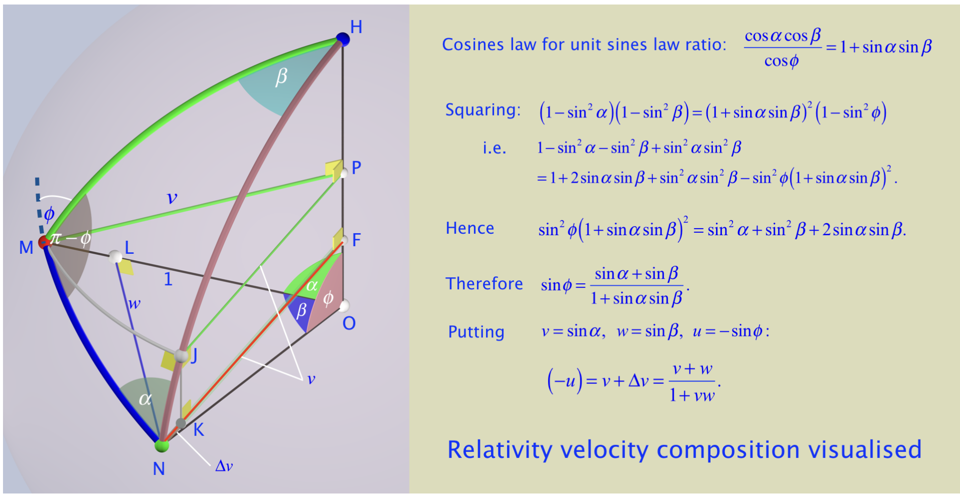

Fig. 1. Relativistic velocity compostion

1. Unit sines law ratio—‘relativistic’—spherical triangles

10th century Persian mathematician *Abul Wafa Buzjani*[a] established the *spherical triangles’ sines law* whereby the ratio of the sines of *surface* angles $\hat{\alpha}$, $\hat{\beta}$, $\hat{\phi}$ to the sines of their respective opposite *centre* angles α, β, ϕ, is the same for all three angles. The *spherical triangles’ cosines law* for a surface angle $\hat{\phi}$

$$\cos\phi = \sin\alpha\sin\beta\cos\hat{\phi} + \cos\alpha\cos\beta\,, \tag{1}$$

was not formulated and proven until Johannes Müller—*Regiomontanus*—wrote his classic *De Triangulis Omnimodis*[1] in 1464.[b]

As easily shown, no exclusively acute spherical triangle can have unit sine ratio angles. If however, as on Fig. 1’s *unit radius sphere*, one surface angle is the *supplement* of its opposite centre angle e.g. $\hat{\phi} = \pi - \phi$ with $\hat{\alpha} = \alpha$ and $\hat{\beta} = \beta$, the sines ratio remains plus one i.e. $\sin\hat{\phi} = \sin(\pi-\phi) = \sin\phi$ but $\cos\hat{\phi} = \cos(\pi-\phi) = -\cos\phi$ and cosines law (1) becomes, after dividing across by $\cos\phi$:

$$\frac{\cos\alpha\cos\beta}{\cos\phi} = 1 - \sin\alpha\sin\beta\left(\frac{-\cos\phi}{\cos\phi}\right) = 1 + \sin\alpha\sin\beta. \tag{2}$$

If angles α, β, ϕ are all acute, as shown in Fig. 1, (2) rearranges as:

$$\sin\phi = \frac{\sin\alpha + \sin\beta}{1 + \sin\alpha\sin\beta}. \tag{3}$$

[a]Discoverer of the identity $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
[b]The German astronomer’s work was published in Nuremberg 1533, almost six decades after his murder in Rome in 1476.

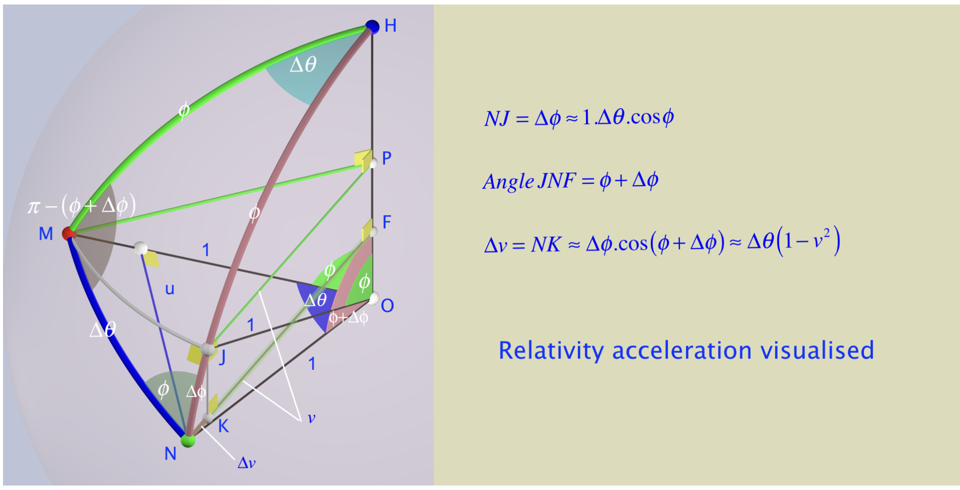

Fig. 2. Relativity acceleration visualized.

Reflecting the *Loedel chart*[2], we substitute $v = \sin\alpha$, $w = \sin\beta$ and[c] $(-u) = \sin\phi$, to obtain the familiar *relativistic velocity composition equation*—scaled for $c = 1$:

$$(-u) = \frac{v+w}{1+vw}. \tag{4}$$

2. The differential relativity spherical triangle

[3],[4]With distances scaled by α/c^2 and times by α/c, where α (now reallocated differently) is the fixed rocket's unscaled own- ('proper') acceleration, we now relabel Fig. 1's previous centre angles α, β and ϕ in Fig. 2 as a unit thrust rocket's *initial*, *differential* and *final* velocity angles $\phi, \Delta\theta$ and $\phi + \Delta\phi$ respectively with $v = \sin\phi$, over a minuscule rocket own-time period $\Delta\tau$. Of special interest are *differential spherical triangles* as $\Delta\tau$ and accordingly angles $\Delta\theta$ and $\Delta\phi$ *all together tend towards zero*. [d]The rocket's differential velocity Δv is obtained by rotating point M about axis OH through 'door' angle $\Delta\theta$ to point J on arc HJN, and dropping a perpendicular onto point K on perpendicular NKF which equals velocity $v + \Delta v$. From the geometry,[e] we have for incremental velocity Δv (segment NK):

$$\Delta v \approx 1.\Delta\theta\cos\phi\cos(\phi+\Delta\phi) \approx \Delta\theta(1-\sin^2\phi) = \Delta\theta(1-v^2) \text{ i.e.}$$

$$\frac{dv}{d\theta} = 1 - v^2. \tag{5}$$

[c]For convenience velocities v, w and u are considered *cyclic*. Hence if v and w are positive, u will be negative and the resulting *forward velocity* is $(-u)$.

[d]A tangent to arc HJN at vertex N is perpendicular to radius NO. Also line NKF is perpendicular to HO. Hence angles JNK and HON both equal $\phi + \Delta\phi$.

[e]$NJ \approx MN\cos\phi$, $NK \approx NJ\cos(\phi+\Delta\phi)$ etc.

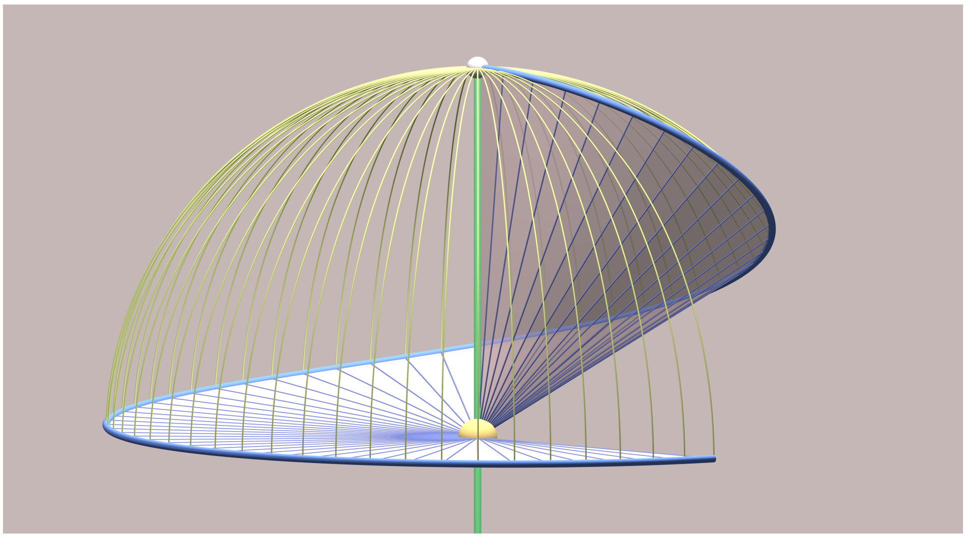

Fig. 3. Differential reference frame angle arcs forming a spiral 'hemix'.

As $\Delta v = 1.\Delta\tau$, velocity composition $v + \Delta v \approx \frac{v+1.\Delta\tau}{1+v.1.\Delta\tau}$ also yields

$$\frac{dv}{d\tau} = 1 - v^2. \tag{6}$$

In the limit as $\Delta\tau$, $\Delta\theta$ and $\Delta\phi$ tend towards zero, a spherical triangles' lateral arc increment $d\theta$ exactly represents a unit thrust rocket's own-time differential $d\tau$.

3. The hemix: an 'own-history-line'

[5]Considering an infinitude of cascaded differential spherical triangles as angle increments become ever smaller, differential lateral arc segments corresponding to rocket clock time intervals form a smooth continuous spherical curve as in Figure 3. The rocket's accumulated own-clock time $\tau = \int_0^\tau d\tau$ equals the curve's path length as well as its 'swept longitude'—by virtue of equations (5) and (6). We label this unique curve *'the hemix own-history-line'* and assign it a special symbol: $\mathfrak{H}$.

$$\text{HEMIX } \mathfrak{H} = {}_{Sph}[1, \tau, \phi] = {}_{Cyl}[v, \tau, \frac{1}{\gamma}] = {}_{Cyl}[\tanh\tau, \tau, \frac{1}{\cosh\tau}] = {}_{xyz}[\tanh\tau\cos\tau, \tanh\tau\sin\tau, \frac{1}{\cosh\tau}]. \tag{7}$$

[f] Although resembling *Nuñes* 1537 loxodromes, curve $\mathfrak{H}$ seems to be absent in the literature e.g. *Davies*' prolonged two-part 1834 treatise[6] on spherical curves, *Yates* 1947 *Handbook on Curves*[7] and the *Encyclopédie des Formes Mathematique Remarquables*—http://www.mathcurve.com.

[f]Cylindrical radius r also equals home frame velocity i.e. $r = \sin\phi = v = \tanh\tau$ and elevation z equals its colatitude's cosine i.e. $z = \cos\phi = 1/\gamma = 1/\cosh\tau$. Rocket own-time τ equals curve path length and traversed longitude θ. This hemispherical curve has several further properties of mathematical interest described in[5].

4. Arbitrary fixed own-acceleration relationships

The familiar standard equations relating home frame time and distance of a rocket accelerating at fixed own-thrust α with its ('proper') own-time τ, are—*with times scaled for unit limit speed*:

$$\alpha t = \sinh \alpha\tau, \quad v = \tanh \alpha\tau, \quad \gamma = \cosh \alpha\tau, \quad \alpha x = \cosh \alpha\tau - 1. \tag{8}$$

Identity $\cosh^2 \alpha\tau - \sinh^2 \alpha\tau \equiv 1$ gives *the hyperbolic worldline* equation:

$$(\alpha x + 1)^2 - (\alpha t)^2 = 1. \tag{9}$$

Equations (8) prompt us to define a *scaled hemix* on a radius $1/\alpha$ hemisphere tracing a fixed thrust rocket's own-time τ proportionately to $\alpha\tau$:

$$\text{Scaled Hemix } \mathfrak{H}_\alpha = {}_{Sph}[\frac{1}{\alpha}, \alpha\tau, \phi] = {}_{xyz}[\tanh \alpha\tau \cos \alpha\tau, \tanh \alpha\tau \sin \alpha\tau, \frac{1}{\cosh \alpha\tau}]/\alpha. \tag{10}$$

4.1. *Radar interval derivations*

As described in[8], we consider home frame observer clocks and rocket clocks synchronised as the rockets are launched together a distance L apart and accelerate at generally *different* fixed own-thrusts ('proper accelerations'). Represented in a single 'home' reference frame's spacetime chart as in Fig. 4, the rear rocket with own-thrust α_r has its launch event situated at the chart's origin $[0, 0]$. In accordance with equations (8) and (9), its home frame hyperbolic worldline coordinates are

$$[x_r, t_r] = \left[\frac{\cosh \alpha_r\tau_r - 1}{\alpha_r}, \frac{\sinh \alpha_r\tau_r}{\alpha_r}\right]. \tag{11}$$

A front rocket with arbitrary fixed own-thrust α_f has its launch event at $[L, 0]$ and hyperbolic worldline coordinates

$$[L + x_f, t_f] = \left[L + \frac{\cosh \alpha_f\tau_f - 1}{\alpha_f}, \frac{\sinh \alpha_f\tau_f}{\alpha_f}\right]. \tag{12}$$

A 'radar photon' emitted at rear rocket's own-time $\acute{\tau}_r = \frac{\sinh^{-1} \alpha_r\acute{t}_r}{\alpha_r}$ has emission coordinate $\acute{x}_r = \frac{\cosh \alpha_r\acute{\tau}_r - 1}{\alpha_r}$. Reflected by the front rocket at its own-time $\hat{\tau}_f = \frac{\sinh^{-1} \alpha_f\hat{t}_f}{\alpha_f}$ and reflection coordinate $L + \hat{x}_f = L + \frac{\cosh \alpha_f\hat{\tau}_f - 1}{\alpha_f}$, the photon travels at unit limit speed in the home frame. Hence $(L + \hat{x}_f) - \acute{x}_r = \hat{t}_f - \acute{t}_r$ i.e.

$$\left(L + \frac{\cosh \alpha_f\hat{\tau}_f - 1}{\alpha_f}\right) - \left(\frac{\cosh \alpha_r\acute{\tau}_r - 1}{\alpha_r}\right) = \frac{\sinh \alpha_f\hat{\tau}_f}{\alpha_f} - \frac{\sinh \alpha_r\acute{\tau}_r}{\alpha_r}.$$

This simplifies as the *general forward radar transit equation*:

$$\frac{e^{-\alpha_f\hat{\tau}}}{\alpha_f} = \frac{e^{-\alpha_r\acute{\tau}}}{\alpha_r} + \frac{1}{\alpha_f} - \frac{1}{\alpha_r} - L. \tag{13}$$

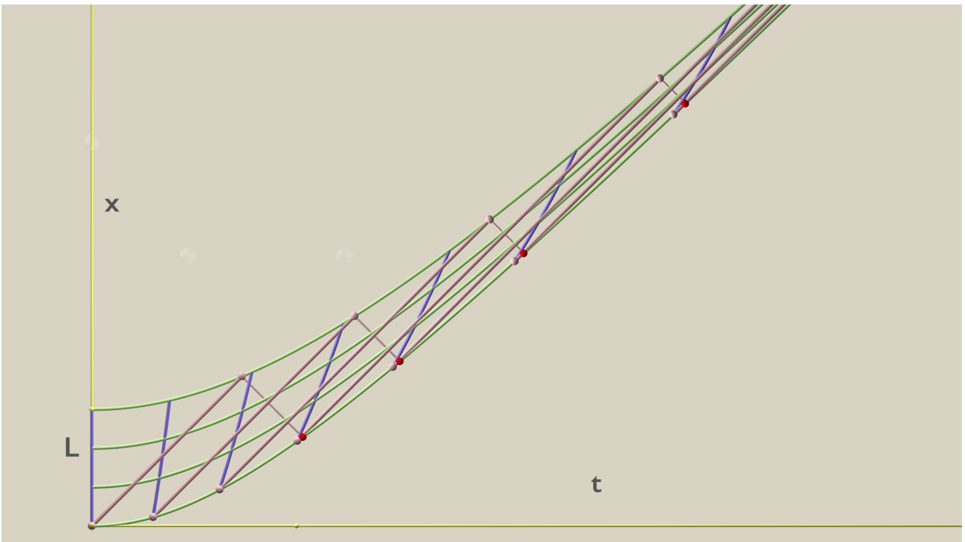

Fig. 4. Home frame world-surface of a 'rigor mortis' acceleration medium with diagonal radar trajectories and curved fixed velocity loci.

The reflected photon meets rear rocket r at home time $\grave{t}_r = \frac{\sinh \alpha_r \grave{\tau}_r}{\alpha_r}$ and home frame *position* $\grave{x}_r = \frac{\cosh \alpha_r \grave{\tau}_r - 1}{\alpha_r}$, over equal home-frame distance and time intervals $(L + \hat{x}_f) - \grave{x}_r = \grave{t}_r - \hat{t}_f$:

$$L + \frac{\cosh \alpha_f \hat{\tau} - 1}{\alpha_f} - \frac{\cosh \alpha_r \grave{\tau}_r - 1}{\alpha_r} = \frac{\sinh \alpha_r \grave{\tau}_r}{\alpha_r} - \frac{\sinh \alpha_f \hat{\tau}}{\alpha_f}.$$

This yields the *general reverse radar transit equation*:

$$\frac{e^{\alpha_f \hat{\tau}}}{\alpha_f} = \frac{e^{\alpha_r \grave{\tau}}}{\alpha_r} + \frac{1}{\alpha_f} - \frac{1}{\alpha_r} - L. \tag{14}$$

Imagining this photon to itself be reflected again i.e *re-emitted* forwards towards the front rocket, by replacing $\acute{\tau}$ with $\grave{\tau}$ and $\hat{\tau}$ with $\check{\tau}$ in (13) we obtain for the front rocket *re-reflection* time $\check{\tau}$:

$$\frac{e^{-\alpha_f \check{\tau}}}{\alpha_f} = \frac{e^{-\alpha_r \grave{\tau}}}{\alpha_r} + \frac{1}{\alpha_f} - \frac{1}{\alpha_r} - L. \tag{15}$$

(13) and (14) yield *the forward general fixed thrust rockets' radar equation*:

$$\frac{1}{{\alpha_f}^2} = \left[\frac{e^{-\acute{\tau}\alpha_r}}{\alpha_r} + \frac{1}{\alpha_f} - \frac{1}{\alpha_r} - L\right]\left[\frac{e^{\grave{\tau}\alpha_r}}{\alpha_r} + \frac{1}{\alpha_f} - \frac{1}{\alpha_r} - L\right]. \tag{16}$$

(14) and (15) give us *the reverse general fixed thrust rockets' radar equation*:

$$\left[\frac{e^{\hat{\tau}\alpha_f}}{\alpha_f} - \frac{1}{\alpha_f} + \frac{1}{\alpha_r} + L\right]\left[\frac{e^{-\check{\tau}\alpha_f}}{\alpha_f} - \frac{1}{\alpha_f} + \frac{1}{\alpha_r} + L\right] = \frac{1}{{\alpha_r}^2}. \tag{17}$$

5. The 'rigor mortis' accelerating medium—'Born rigidity'

5.1. *Rigor mortis radar intervals*

[g] [9]The constant parameter terms in relationships (13)–(17) are eliminated by applying what we call *the 'rigor mortis' condition*[h]

$$L = \frac{1}{\alpha_f} - \frac{1}{\alpha_r} \text{ i.e. } \alpha_f = \frac{\alpha_r}{1 + L\alpha_r} \text{ and } \alpha_r = \frac{\alpha_f}{1 - L\alpha_f}. \tag{18}$$

Forward radar equation (16) $e^{(\grave{\tau}-\acute{\tau})\alpha_r} = \frac{{\alpha_r}^2}{{\alpha_f}^2} = (1 + L\alpha_r)^2$ then yields *the constant forward 'rigor mortis' radar interval:*

$$\grave{\tau} - \acute{\tau} = \frac{2}{\alpha_r} \ln(1 + L\alpha_r) = \frac{2}{\alpha_r} \ln(\frac{\alpha_r}{\alpha_f}). \tag{19}$$

Also from reverse radar equation (17) $e^{-(\check{\tau}-\hat{\tau})\alpha_f} = \frac{{\alpha_f}^2}{{\alpha_r}^2} = (1 - L\alpha_f)^2$, a second value emerges—*the constant reverse 'rigor mortis' radar interval*:

$$\check{\tau} - \hat{\tau} = -\frac{2}{\alpha_f} \ln(1 - L\alpha_f) = \frac{2}{\alpha_f} \ln(\frac{\alpha_r}{\alpha_f}). \tag{20}$$

$$\text{Moreover} \qquad \frac{\grave{\tau} - \acute{\tau}}{\check{\tau} - \hat{\tau}} = \frac{\alpha_f}{\alpha_r} = \frac{1}{1 + L\alpha_r} = 1 - L\alpha_f. \tag{21}$$

The 'rigor mortis' forward and reverse radar intervals are constant and their ratio equals the rocket accelerations' ratio α_f/α_r.

5.2. *Rigor mortis space and time dispersals*

For any set $v = \tanh(\tau_f \alpha_f) = \tanh \tau_r$ and $\gamma = \cosh(\tau\alpha_f) = \cosh \tau_r$, (11), (12) and (18) yield the rockets' home frame *time dispersal*

$$t_f - t_r = \frac{\sinh(\tau_f \alpha_f)}{\alpha_f} - \frac{\sinh \tau_r}{1} = \gamma v \left[\frac{1}{\alpha_f} - 1\right] = \gamma v L,$$

and their home frame *distance dispersal*

$$(x_f + L) - x_r = \frac{\gamma - 1}{\alpha_f} + L - \gamma + 1 = \gamma \left[\frac{1}{\alpha_f} - 1\right] - \frac{1}{\alpha_f} + L + 1 = \gamma L.$$

From the (Larmor-)Lorentz transformation for *the corresponding comoving frame*, time dispersal turns out to be permanently zero.[i]

$$\Delta\tau = \gamma \left[(t_f - t_r) - v(x_f + L - x_r)\right] = \gamma \left[\gamma v L - v\gamma L\right] = 0. \tag{22}$$

[g]The commonly used term *'rigid motion'* is anachronistic since it has *a different meaning* in differential geometry.
[h]This relationship was established in an elaborate manner in 2003 by *Woodhouse* [10] (p.115).
[i](22) and (23) were derived in 2010 as solutions to a set of mathematical equations by *Franklin* [11].

Distance dispersal remains equal to the rockets' launch separation:

$$\gamma\left[(x_f + L - x_r) - v(t_f - t_r)\right] = \gamma\left[\gamma L - v^2\gamma L\right] = L. \tag{23}$$

5.3. *The 'rigor mortis' home frame world-surface*

We now rescale times and lengths so that rear rocket own-thrust α_r as well as c are *one*. Fig. 4 shows home frame rigor mortis worldlines not only for the rear and front rockets but also for intermediate medium increments identified by their launch distance l from the rear rocket l_0 and assumed to have their own 'minuscule rockets' as well as—'in the limit'—zero mass. Hence no inter-increment forces or delays are entailed. Each arbitrary medium increment l then accelerates at $\alpha = 1/(1+l)$ (equation (18)) with its worldline elevated from start at $x_0 = l$.

Also shown are fixed velocity loci which share comoving inertial frames. Emitted and reflected radar trajectories appear as diagonal lines corresponding to scaled $c = \pm 1$. Substituting $L = 0.57$, $a_r = 1$ and $a_f = 1/(1+L) = 0.637$ in equation (19) and dividing by the chosen rocket own-time period $3\pi/32$ between respective emissions, yields 3.063. This clearly corresponds to each of the chart's radar response intervals in terms of the rear rocket's own-time emission interval.

5.4. *The rigor mortis real-metric own-surface*

Fig. 5 portrays a 3-dimensional 'real-metric own-surface' hosting medium curves which share identical home frame velocities and whose total metric length remains unchanged in corresponding ever changing *inertial* frames by virtue of rigor mortis condition (18). These are crossed by scaled hemix curves tracing respective increment own-times. The unit thrust rear rocket's increment curve is represented by equation (7)'s hemix on the inner unit radius hemisphere. The front rocket's hemix traverses the outer hemisphere whose radius is $\frac{1}{\alpha_f} = 1+L$ in accordance with rigor mortis condition (18) and scaled hemix equation (10). Intermediate increment hemices are described by the same equations.

For shared home frame velocities, each hemix metrically traces its respective fixed α-thrust increment's elapsed own-time since launch: $\tau = \tau_r/\alpha = \tau_r(1+l)$. The more slowly accelerating increments nearer the front rocket require greater clock own-times τ than those nearer rear rocket to attain the same shared home frame velocity $v = \tanh\tau_r = \tanh(\tau\alpha) = \tanh(\tau/(1+l))$, and so be relatively stationary to one another in each comoving inertial frame. In contrast to Fig. 4's shared *curved* velocity loci, in Fig. 5 such loci appear as constant length L *straight lines* radially distributed at colatitude angle $\phi = \sin^{-1} v$ and spread along the surface at rear increment own-time intervals $\Delta\tau_r = 3\pi/32$ for $0 \le \tau_r \le 3\pi/2$.

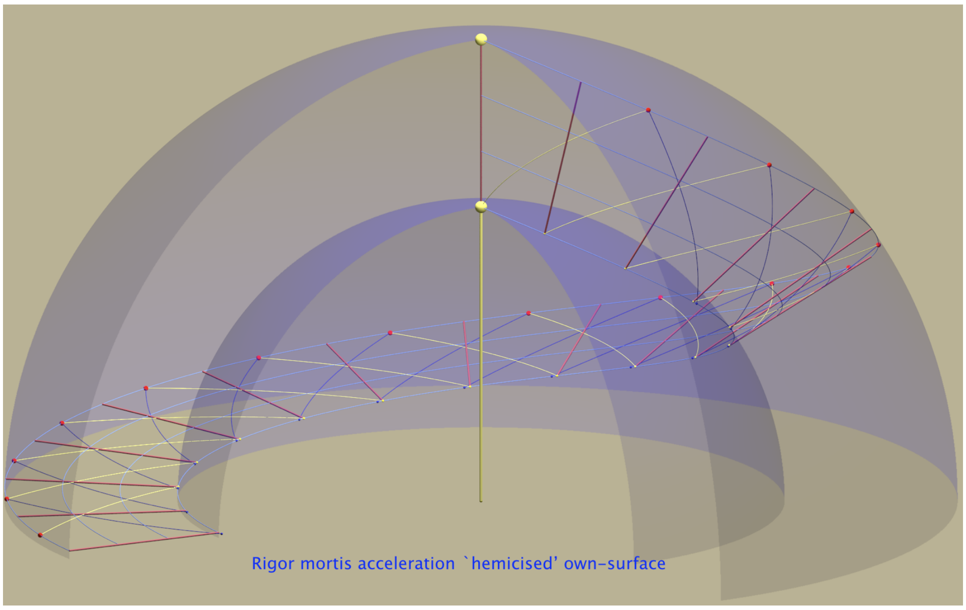

Fig. 5. The 'hemicised' rigor mortis own-surface

Radar trajectory expressions for respective rear rocket emission own-times $\acute{\tau}$ are obtained by combining forward transit equation (13) which reduces to $e^{-(1+l)\tau} = e^{-\acute{\tau}}(1+l)$, and equation (10). Crucially, the repeated forward radar period $3.063 \cdot \frac{3\pi}{32}$ is *metrically exactly manifest* on the rigor mortis own-surface.

6. The rigor mortis real-metric equation

By virtue of the rigor mortis own-surface's spherically symmetric hemix segments and its constant length radial medium curves, as a ruled surface it may be transformed to a flat surface as in Fig. 6 without intrinsic surface metric distortions. Adopting Figure 6's planar surface's centre as its origin and using radial coordinates (r, θ), we may therefore write $r = 1 + l$ and $\theta = \tau_r = \tau/(1 + l)$. A surface's 'differential metric' relates any two minimally apart event points—in the limit.

As $dr = dl$ and $d\theta = d\tau/(1 + l)$ and our flat surface's metric interval is $ds^2 = dr^2 + r^2 d\theta^2$ i.e. $ds^2 = dl^2 + (1 + l)^2 d\tau^2/(1 + l)^2$, we obtain for the general length parameter λ which in this special rigor mortis case happens to equal l:

The rigor mortis medium's own-surface real-metric
$$ds^2_{\mathfrak{RM}} = d\tau^2 + d\lambda^2. \tag{24}$$

Since all distances and surface angles of the surface's medium and increment curves remain unchanged, both surfaces are intrinsically the same i.e. *'isometric'*. Hence metric (24) applies to both the planar own-surface as well as to the 'hemicised' own-surface.

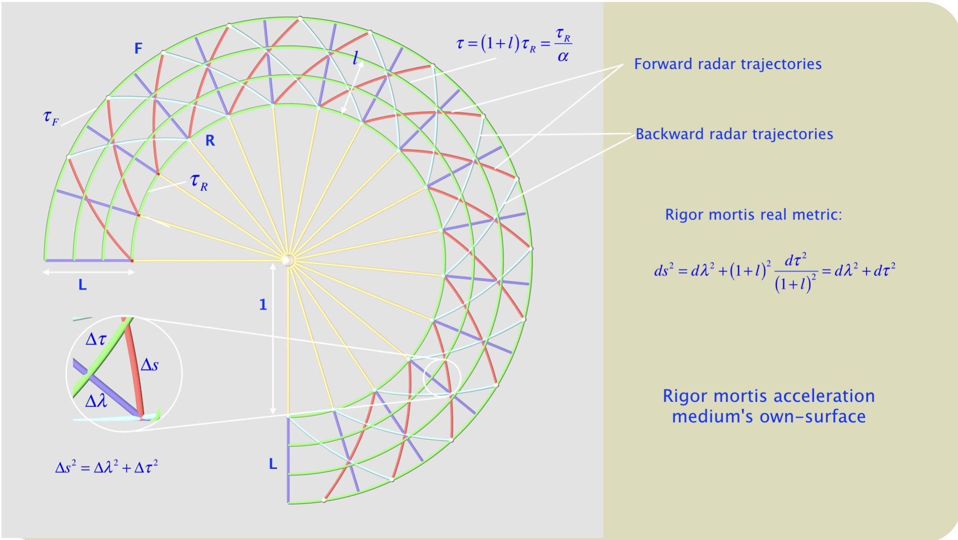

Fig. 6. The rigor mortis flat own-surface

7. A one-off extended medium vindication of Minkowski's metric

If we replace the *positive* sign in our *real variables* rigor mortis own-surface metric (24) by a *negative* sign, we obtain the *equivalent* complex variable Minkowski spacetime metric interval equation:[j]

$$ds^2_{\mathfrak{RM}} = d\tau^2 - d\lambda^2. \tag{25}$$

Our real metric surface approach has resolved the rigor mortis topic at a quite elementary level which nevertheless reflects several advanced differential geometry concepts familiar to relativists used to dealing with the issue in the traditional yet more obscure Minkowski spacetime approach which, as argued in the following, is otherwise inapplicable to all *other* extended medium acceleration scenarios.

8. Homogeneous acceleration—Bell's string paradox

8.1. *Inter-rocket radar intervals*

If $\alpha_f = \alpha_r = \alpha = 1$, time and lengths being *rescaled* so that α as well as c are *one*, forward and reverse radar equations (16) and (17) respectively reduce to:

$$e^{\check{\tau}} = \frac{1}{[e^{-\acute{\tau}} - L]} + L \text{ and } e^{\check{\tau}} = 1/\left[\frac{1}{[e^{\hat{\tau}} + L]} - L\right] = \left[\frac{[e^{\hat{\tau}} + L]}{1 - L(L + e^{\hat{\tau}})}\right] \tag{26}$$

[j]In its one spatial dimensional form. Sometimes written as $ds^2 = d\lambda^2 - d\tau^2$.

So for $\acute{\tau} < \ln(1/L)$, THE UNIT ACCELERATION REAR ROCKET'S RADAR INTERVAL

$$\grave{\tau} - \acute{\tau} = \ln\left[\frac{1}{e^{-\acute{\tau}} - L} + L\right] - \acute{\tau}. \tag{27}$$

Likewise (from (26)-ii) THE FRONT ROCKET'S RADAR INTERVAL

$$\check{\tau} - \hat{\tau} = \ln\left[\frac{1 + Le^{-\hat{\tau}}}{1 - L(L + e^{\hat{\tau}})}\right]. \tag{28}$$

As not generally appreciated,[k] 'radar distance' between identically accelerating rockets varies i.e. THE SECOND POSTULATE DOES NOT APPLY FOR EXTENDED OBJECTS UNDER IDENTICAL FIXED OWN ('PROPER') ACCELERATION.

8.2. *The homogeneously accelerating medium in the home frame*

For an idealised 'massless' medium between the rockets with each part accelerating with the same identical unit thrust as the two rockets themselves, equation (9) gives THE UNIT THRUST MEDIUM'S HOME FRAME WORLD-SURFACE EQUATION

$$(x - l + 1)^2 - t^2 = 1. \tag{29}$$

The medium's hyperbolic worldlines are shown in Figure 7, with trajectories of photons emitted from the rear rocket and reflected back from the identically accelerating front rocket. The vertical lines represent the medium itself at equal *rocket own-time* intervals $\Delta\tau$. Substitution of $L = 0.5548$, $\Delta\tau = 3\pi/32$, $\acute{\tau}_0 = 0$ and $\acute{\tau}_1 = 3\pi/32$ in radar equation (27) yields: $\grave{\tau}_0 - \acute{\tau}_0 = \frac{3\pi}{32} \cdot 3.497$ and $\grave{\tau}_1 - \acute{\tau}_1 = \frac{3\pi}{32} \cdot 4.977$. These intervals correspond to those in the computer generated diagram where emitted and reflected photon trajectories are *straightline* $\pm 45°$ *diagonals* (just as in Figure 4). The fixed velocity loci in this case are just straight lines.

9. Radar mappings from inertial home frame onto own-surface Υ

Figure 8 shows a one-to-one 'homeomorphic' mapping of Figure 7's curves onto a unit pitch helicoidal own-surface generated by hemix curve $\mathfrak{H}$, as established in[3] and also discussed earlier[l] at[13]:

$$\text{HEMICOID } \Upsilon(\tau, l) = \left[\tanh\tau\cos(\tau + l), \tanh\tau\sin(\tau + l), \frac{1}{\cosh\tau} + l\right]. \tag{30}$$

[k] An extraordinary example of this 'historic' misconception is the much cited AJP 1987 paper[12].
[l] *Deutsche Physikalische Gesellschaft* 2012 Spring Conference in Göttingen: http://www.dpg-verhandlungen.de/year/2012/conference/goettingen/part/gr/session/4/contribution/4 .

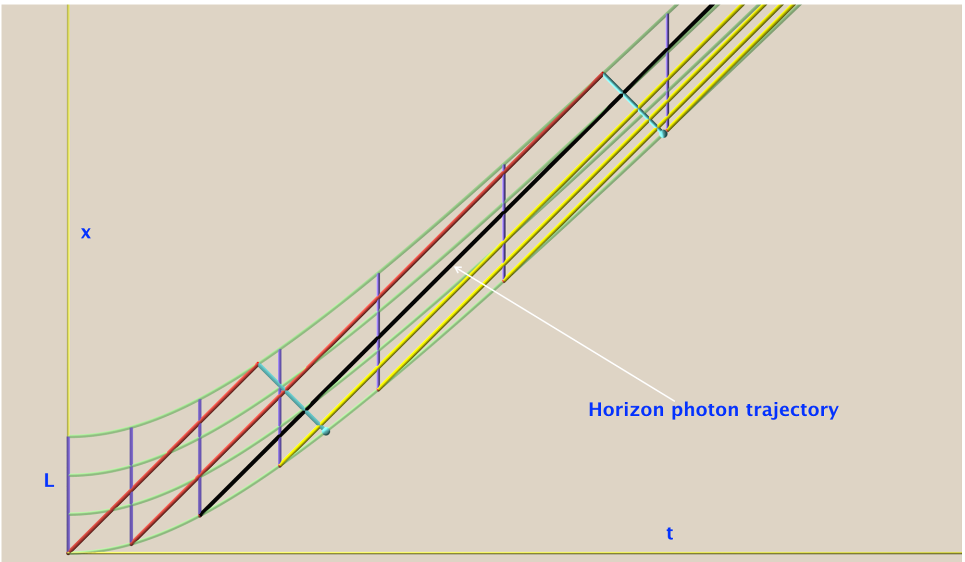

Fig. 7. Home frame world-surface of a homogeneously accelerating medium, with reflected and nonreflected radar trajectories and fixed velocity loci.

9.1. *Outgoing photon paths*

For *an outgoing photon's radar equation* ρ, we turn to radar transit equation (13) for an arbitrary photon's rear rocket emission own-time $\acute{\tau}$, with $\alpha_f = \alpha_r = 1$ and replace $\hat{\tau}$ with τ and l with $-e^{-\tau} + e^{-\acute{\tau}}$ in (30):

$$\rho = \left[\tanh\tau\cos(\tau - e^{-\tau} + e^{-\acute{\tau}}), \tanh\tau\sin(\tau - e^{-\tau} + e^{-\acute{\tau}}), \frac{1}{\cosh\tau} - e^{-\tau} + e^{-\acute{\tau}}\right]. \tag{31}$$

For the chosen values of $L = 0.5548$ and $\Delta\tau = \frac{3\pi}{32}$, only the first two photons emitted are reflected. From forward radar transit equation (13) $e^{\hat{\tau}} = 1/(e^{-\acute{\tau}} - L)$ so for a photon whose emission time is $\acute{\tau} = \ln(1/L)$, $e^{\hat{\tau}} = \infty$. As a medium's shared own-time τ approaches ∞, the 'horizon photon' trajectory (in black) thus tends to *'surf'* the front rocket i.e. get ever closer to it at nearly zero speed, without ever reaching it. Later photons (coloured yellow) surf respective *intermediate* medium increments. A UNIT THRUST MEDIUM'S REAR ROCKET-EMITTED NONREFLECTED PHOTON TRAJECTORIES 'SURF' INCREMENT CURVES IN THE LIMIT.

9.2. *Returning photon paths*

Using forward forward and returning transit equations (13) and (14) we obtain: $\grave{\tau} - \hat{\tau} = \ln\left[L(e^{-\acute{\tau}} - L) + 1\right]$, if $e^{-\acute{\tau}}$ is very close to L, then $\grave{\tau} - \hat{\tau} \approx \ln(1) = 0$. In such a case, as τ approaches ∞ a *counter-directional* reflected photon would traverse the medium at a virtually infinite 'crossing rate'.

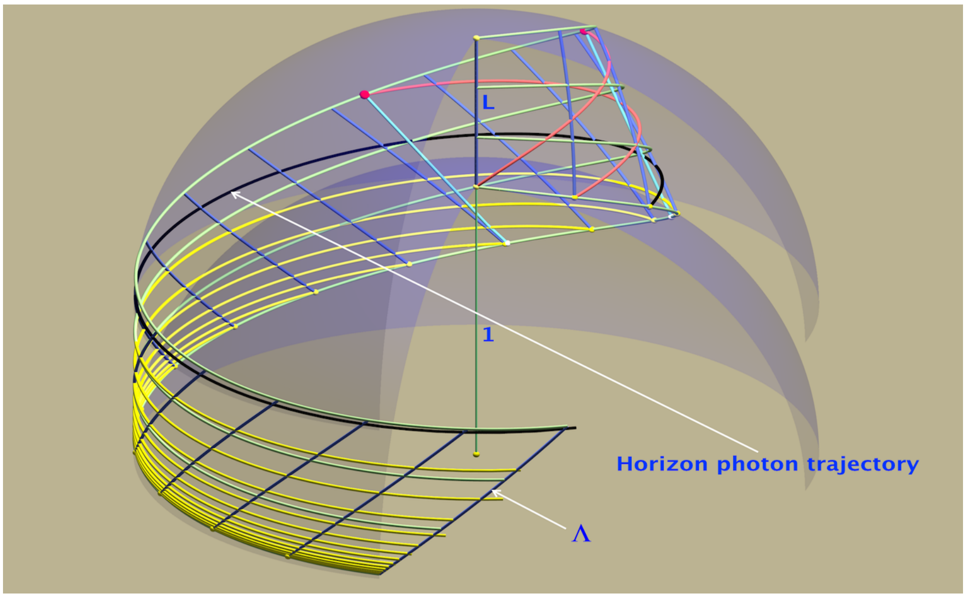

Fig. 8. Own-surface increment and medium curves with radar trajectories

FOR AN EMISSION TIME $\acute{\tau}$ NEAR HORIZON VALUE $\ln 1/L$, A PHOTON'S RETURNING UNIT THRUST MEDIUM'S TRAVERSING TIME TENDS TOWARDS ZERO. A REFLECTED PHOTON WOULD TEND TO CROSS THE ENTIRE MEDIUM AT A 'SUPRALUMINAL' SPEED. By *'supraluminal'* (on top of) we mean almost infinite, as opposed to *'superluminal'*—faster-than-light yet *finite*, speed. We recall that the term 'speed' here relates to *a medium's own-length crossings time-rated by an imagined third party observer recording crossed medium increments' progressing own-time values.* Of course both emitted and reflected photons always propagate at *unit scaled limit speed* in the *inertial* home frame as well as in all other comoving inertial frames.

9.3. *The unit thrust medium's 'noninertial length'*

Medium curves' tangent vector moduli on Υ yield the Bell string's paradox expansion factor (where λ is an increment's *'noninertial length'* from the rear rocket):

$$\epsilon(\tau) = \frac{\partial \lambda}{\partial l} = |\Upsilon_{l}| = \sqrt{\frac{\partial \Upsilon}{\partial l} \cdot \frac{\partial \Upsilon}{\partial l}} = \sqrt{1 + \tanh^2 \tau} = \sqrt{1 + v^2}.$$

The medium's total noninertial own-length is $\Lambda = L.\sqrt{1 + \tanh^2 \tau}$ in its *proxy noninertial frames* $\underline{\Pi}$. As 'viewed'[m] from such frames, as long as the medium keeps accelerating, a uniformly accelerating medium never expands beyond *the square root of two* times its launch length.

[m] *In a manner of speaking*, since there is no 'conventional' way (at least to date) of assessing 'noninertial length'.

9.4. *The uniform thrust medium's real metric*

Unit thrust own-surface (30) can be expressed in cylindrical coordinates as:

$$\Upsilon(\tau, l) = {}_{Cyl}\,[r, \theta, z] = {}_{Cyl}\left[\tanh\tau, \tau + l, \frac{1}{\cosh\tau} + l\right].$$

Total differentials $dr, d\theta$ and dz are easily written:

$$dr = \frac{1}{\cosh^2\tau}d\tau; \qquad d\theta = d\tau + dl; \qquad dz = \frac{-\sinh(\tau)}{\cosh^2\tau}d\tau + dl.$$

As $r = v = \tanh\tau$ and $d\lambda = \sqrt{1+\tanh^2\tau}.dl$, a surface's metric in $\mathbb{R}^3$ being $ds^2_\Upsilon = dr^2 + r^2 d\theta^2 + dz^2$ yields THE UNIT THRUST MEDIUM'S OWN-SURFACE METRIC

$$ds^2_\Upsilon = d\tau^2 + d\lambda^2 + 2\tanh\tau\left(\frac{\tanh\tau - 1/\cosh\tau}{\sqrt{1+\tanh^2\tau}}\right)d\tau.d\lambda. \tag{32}$$

Crucially, this contains a *variable coefficient mixed differentials expression*

$$2\tanh\tau\left(\frac{\tanh\tau - 1/\cosh\tau}{\sqrt{1+\tanh^2\tau}}\right)d\tau.d\lambda.$$

9.5. *Minkowski spacetime's nongenerality in special relativity*

Clearly, unit thrust own-surface Υ properly conforms to a formidable set of conditions conceivable for the homogeneous accelerating medium. Its metric (32) therefore definitely constitutes a solution to the homogeneous acceleration expansion problem. Proponents of Minkowski metric's general validity in special relativity might argue that another actual solution may lie in *pseudo-Euclidean geometry*. After all the rigor mortis' real metric itself may be so transformed just by replacing a real coordinate by an 'equivalent' imaginary one. Others, perhaps acknowledging that the hemicoid surface's properties may well reflect a *coincidental* solution, might insist, as claimed by *Synge* in his well known 1956 classic *Relativity: The Special Theory*[14] (Section 11, *The fundamental quadratic form*, page 17's equation (20)), that *"a non-singular quadratic form"* metric equation *"appropriate to Einstein's [special] relativity"* (such as metric (32), if correct), should be somehow mathematically *equivalent* i.e. *reducible* by a 'coordinate transformation' to the quadratic form $ds^2 = d\tau^2 - d\lambda^2$ (in single spatial dimension metric form). Nevertheless there is no way that metric (32)'s *nonconstant* mixed differential terms coefficient could be cancelled out by some conceivable transformation of coordinates. The nonzero mixed $d\tau.d\lambda$ term is *wholly incompatible* with the Minkowski spacetime interval ${ds_{\mathfrak{M}}}^2 = d\tau^2 - d\lambda^2$.

Where one might expect a cogent case for the obfuscating Minkowski approach to the extended accelerating medium issue to be presented, no consistent arguments are evident, either in textbooks or papers. A 'paramount' book on Minkowski Spacetime is *Naber's* 1992/2010 classic[15] which (pages 2-4) discusses separate "admissible" observers' spacetime frames wherein "photons propagate rectilinearly with [scaled] speed 1". Yet nowhere in Naber's book, nor indeed in Synge's work, is the issue of a homogeneously (or otherwise) accelerating *extended* medium in special relativity properly addressed, a surely curious state of affairs and arguably in itself a firm indication that *Minkowski spacetime is not generally valid in special relativity.* In spite of its intriguing title, *Brown and Pooley's* 2004 paper *Minkowski space-time: a glorious non-entity* [16], is primarily concerned with philosophical matters and does not specifically address noninertial length issues.

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The remainder of this paper discusses further medium acceleration scenarios.

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