Tidal Effects in Gravitation Measurements by Light-Pulse Atom Interferometry

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The acceleration measured by a light-pulse Atom Interferometer (AI) in a non uniform gravitational field systematically deviates from the true acceleration by a term to first order in the gravity gradient. A recent proposal to (ideally) cancel out the gravity gradient by means of an appropriate frequency shift of one laser pulse overcomes the shortcomings of previous proposals based on physical reversal of the instrument axis, and has already been shown to be effective. However, it does not eliminate the deviation. This tidal acceleration error affects the absolute measurement of the local gravitational acceleration at a level that is relevant for the current uncertainty, but it is negligible in drop tests of the Universality of Free Fall with a dual AI as long as they can rely on the same laser frequency to interrogate the different atoms species. In gravity gradiometers based on atom interferometry and used for the measurement of the universal constant of gravity and the detection of gravitational waves the relevance of the deviation needs to be assessed relative to the target of the experiment.

Since the gravitational force is not uniform, gravitation measurements to high precision and accuracy are affected by systematic gravity gradient (tidal) effects. While tidal effects are ubiquitous no matter which kind of test masses and measurement setups are used, the way in which they affect the measurement, and how they can be reduced, can differ significantly. In this letter we focus on tidal effects in gravitation measurements by atom interferometry, referring to similar measurements with bulk masses only for comparison.

In light-pulse Atom Interferometers (AI) (see e.g. [1, 2]) the atom clouds which constitute the test masses are subjected to the gravitational field of the Earth. The equation of motion of a point mass (the center of mass of an atom cloud or that of a bulk mass alike) falling on the surface of the Earth, to first order in the Earth's gravity gradient, reads:

$$\ddot{z} = \frac{GM_\oplus}{(R_\oplus + z)^2} \simeq g_\circ - \gamma z(t) \tag{1}$$

with $M_\oplus, R_\oplus$ the mass and radius of Earth, $g_\circ = \frac{GM_\oplus}{R_\oplus^2}$ the local gravitational acceleration, $\gamma = \frac{2g_\circ}{R_\oplus} \simeq 3.1 \times 10^{-6}\,\mathrm{s}^{-2}$ the Earth's gravity gradient, and the vertical z axis pointing downwards. In a perturbative theory approach the instantaneous position $z(t)$ that multiplies γ must be of zero order in γ, i.e.:

$$z(t)_{|\gamma=0} = z_\circ + v_\circ t + \frac{1}{2} g_\circ t^2 \ . \tag{2}$$

with $z_\circ, v_\circ$ the initial position and velocity of the test mass. Hence, the equation of motion to first order in γ is:

$$\ddot{z} = g_\circ - \gamma \Big(z_\circ + v_\circ t + \frac{1}{2} g_\circ t^2 \Big) \ . \tag{3}$$

With three laser pulses at times $0, T, 2T$ (labeled as 1,2,3) the AI measures the phase difference [1]:

$$\delta\Phi = [\Phi_3 - \Phi_2] - [\Phi_2 - \Phi_1] \tag{4}$$

and if the momentum transfer k is the same in all three pulses, at their respective times, then $\Phi_1 = kz(0)$, $\Phi_2 = kz(T)$, $\Phi_3 = kz(2T)$, hence, by integrating twice the equation of motion (3) in order to obtain $z(t)$ we have:

$$\delta\Phi = kT^2 \Big[g_\circ - \gamma \Big(z_\circ + v_\circ T + \frac{7}{12} g_\circ T^2 \Big) \Big] \tag{5}$$

yielding a value of the local gravitational acceleration to first order in γ, as measured by the AI:

$$g_{_{AI}} = g_\circ - \gamma \Big(z_\circ + v_\circ T + \frac{7}{12} g_\circ T^2 \Big) \ . \tag{6}$$

The phase difference (5) and the resulting acceleration (6) have been derived by various authors [1, 3] based on the tutorial [4], and are generally accepted.

However, by comparing (6) with the obvious equation(3), the coefficient $7/12$ instead of $1/2$ in the $g_\circ T^2$ term is puzzling. Even more so if we look at free-fall absolute gravimeters in which the motion of the test mass (a corner-cube retroreflector) is monitored by laser interferometry. They have achieved the best absolute measurement of the local gravitational acceleration, to 1.1×10^{-9}, have investigated the motion of the test mass very carefully and nowhere in their calculations we see the $7/12$ numerical factor ([5], with Appendix 1; [6]). One may be led to conclude that the discrepancy may be related to the physical nature of the falling mass, hence to a sort of "quantum" versus "classical" approach to the problem. But it is not so. Following a discussion with Neil Ashby, it was concluded and briefly reported in [7] that the $7/12$ factor is simply due to the fact that the AI measures the acceleration from only three position measurements of the atom cloud. .

Consider the instantaneous position $z(t)$ to order γ as obtained from the equation of motion (3):

$$z(t) = z_\circ + v_\circ t + \frac{1}{2} g_\circ t^2 + \gamma \left[\frac{1}{2} z_\circ t^2 + \frac{1}{6} v_\circ t^3 + \frac{1}{24} g_\circ t^4 \right] \tag{7}$$

and imagine to have only three values of it available, at the times of the three laser pulses, i.e. $z(0), z(T), z(2T)$. From these three values we can derive the average velocity $\bar{v}_{0-T}$ in the time interval between the first and the second pulse, and the average velocity $\bar{v}_{T-2T}$ in the next time interval, between the second and the third laser pulse. The change of the mean velocity divided by T yields the best approximation to the actual value of the acceleration obtainable from the three positions $z(0), z(T), z(2T)$, and that turns out to be g_{AI} with the 7/12 coefficient in the last term:

$$g_{approx} = \frac{\bar{v}_{T-2T} - \bar{v}_{0-T}}{T} = \frac{1}{T}\left[\frac{z(2T)-z(T)}{T} - \frac{z(T)-z(0)}{T}\right] = g_{AI} \ . \tag{8}$$

Being an approximation to the exact (to order γ) acceleration (3), this result does not completely agree with it. The discrepancy concerns the acceleration term which is quadratic in time. Since it comes from the position term containing the time to power four, it is not surprising that it cannot be recovered correctly from only three position measurements. The argument holds whatever the nature of the freely falling test mass.

It follows that, as reported in [7], the acceleration measured by the light-pulse AI is systematically incorrect by the amount

$$\Delta g_{AI} = \frac{1}{12}\gamma g_\circ T^2 \ . \tag{9}$$

It has been proposed by [8] that the momentum transfer k_2 of the second laser pulse (at time T) be made different from the value k applied by the first and third pulses by an amount Δk chosen as follows:

$$k_2 = k + \Delta k = k - \frac{1}{2}\gamma T^2 k \ . \tag{10}$$

In this case the phase difference, to order γ, measured by the interferometer reads:

$$\begin{aligned} \delta\Phi_{\Delta k} = \\ kT^2\Big[g_\circ - \gamma\Big(z_\circ + v_\circ T + \frac{7}{12}g_\circ T^2\Big)\Big] + \\ -2\Delta k\Big[\Big(z_\circ + v_\circ T + \frac{1}{2}g_\circ T^2\Big) + \\ -\gamma T^2\Big(\frac{1}{2}z_\circ + \frac{1}{6}v_\circ T + \frac{1}{24}g_\circ T^2\Big)\Big] \end{aligned} \tag{11}$$

and, if k_2 is exactly as stated in (10), it reduces to:

$$\delta\Phi_{\Delta k} = kT^2\Big(g_\circ - \frac{1}{12}\gamma g_\circ T^2\Big) \tag{12}$$

showing that the tidal acceleration terms $\gamma(z_\circ + v_\circ T)$, proportional to the initial position and velocity of the atom cloud have been cancelled, but the tidal term (9) remains, as pointed out by [9].

This fact can be easily explained based on the previous derivation of (9) and by looking at Fig. 2 of [8]. This figure shows that the effect of tides is to open up the trajectories of the two branches of the atom interferometer, unlike in the presence of a uniform field. Thus, the idea is to apply an appropriate Δk at the second laser pulse so as to close the trajectories, as if the atom cloud were falling in a uniform field. However, the value of Δk which makes this scheme to work is established on the basis of the true motion of the atom cloud, and this motion obeys equation (3); to the contrary, according to the phase difference measured by the interferometer, it moves with the acceleration (6). The result is a discrepancy expressed by the acceleration term (9), as found by [9].

Since this term originates from an incorrect (approximated) measurement of the acceleration by the atom interferometer, it should be dealt with very carefully whenever it is found to be relevant for the gravitation measurement of interest.

The absolute measurement of g performed by [10, 11] to a relative uncertainty of about 3×10^{-9} is one such case. At this level systematic errors to first order in γ, assumed to be given as in (6), are relevant, and very careful systematic checks have been performed in order to model them and reduce their contribution to the overall systematic error of the measurement. With $T = 160\,\mathrm{ms}$, the unmodelled error (9) contributes with an uncertainty of

$$\frac{\Delta g_{AI}}{g_\circ} \simeq \frac{1}{12}\gamma T^2 \simeq 6.6 \times 10^{-9} \tag{13}$$

and would therefore require attention. Were the technique [8] implemented in this case to cure tidal effects related to the initial conditions, this error would remain and although it could be modelled, it is quite intriguing that the implementation itself would be based on phase measurements affected by this very same error.

It is also worth stressing that whenever the error (9) matters, attempts at increasing the time T in order to increase the sensitivity of the interferometer (the phase difference grows as T^2) results in increasing this error in the measured phase difference as T^4, and therefore as T^2 in relative terms.

The main motivation of [8] was to improve the performance of cold atom drop tests of the Universality of Free Fall (UFF) and the Weak Equivalence Principle (WEP) by mitigating the limitations imposed by tidal effects ([2, 7]). These tests are performed by dropping two clouds of different atoms (or just different isotopes) A and B in a dual atom interferometer, and measuring the phase difference for each species, the physical observable being the difference between the two. The preferred choice for the atom clouds is ^{87}Rb and ^{85}Rb, because the small isotope shift makes it possible to use the same laser pulses to simultaneously manipulate the two clouds, thus avoiding time synchronization errors. This is certainly very helpful from the experimental point of view, but limits the choice of atoms that can be tested.

By applying the proposal [8] with Δk exactly as required by (10) we have:

$$\begin{aligned}\delta\Phi^A_{\Delta k} - \delta\Phi^B_{\Delta k} = \\ kT^2\Big[\Big(g^A_\circ - g^B_\circ\Big) - \frac{1}{12}\gamma T^2\Big(g^A_\circ - g^B_\circ\Big)\Big] \ .\end{aligned} \quad (14)$$

The first term yields the Eötvös parameter $\eta = \frac{g^A_\circ - g^B_\circ}{g_\circ}$ which quantifies the level of UFF-WEP violation and is exactly zero if there is no violation; the remaining tidal term, resulting from (9), is a factor

$$f = \frac{1}{12}\gamma T^2 \simeq 2.6\cdot 10^{-7} T^2 \quad (15)$$

smaller than η and negligible, as stated by [7] and [9]. The proposal [8] is therefore worth implementing.

A previous proposal to reduce gravity gradient effects in this kind of experiments was based on the idea of rotating the instrument axis [12]. In a data set of $10 + 10$ drops (10 in one direction and 10 with a reversed axis) the contribution from gravity gradient would, ideally, cancel out. The numerical simulations carried out by the proposers report a reduction of the gravity gradient effects by many orders of magnitude. However, for axis reversal to work the initial offset vectors between the two atom clouds must follow the axis reversal of the instrument in all drops, that is, the cloud which at the initial time was closer to Earth, must be farther from Earth in the corresponding drop with the instrument axis reversed. Indeed, the simulations assume that "the initial condition mismatches are 100% fixed to the apparatus".

While in space tests of the WEP with bulk masses, such as Microscope and GG [13], it is an obvious assumption that the offset vectors are fixed with the apparatus, this is not at all obvious in mass dropping tests in which a very high number of drops (each one with its own initial conditions) are needed in order to reduce the random measurement noise of the instrument. Being systematic, the tidal acceleration error must be below the target acceleration $a_{target} = \eta g_\circ$ of the test in all drops. If this requires a gravity gradient reduction by a factor k, and if random noise needs a total number of n data sets –each one based on $10 + 10$ drops– to reach a_{target}, should mismatch reversal not occur in just 1 single drop out of the entire measurement, the resulting average acceleration is already larger then the target:

$$\begin{aligned}< a >= \frac{(10n-1)a_{target} + k a_{target}}{10n} = \\ a_{target}\Big(1 + \frac{k-1}{10n}\Big) > a_{target} \ .\end{aligned} \quad (16)$$

The higher the gravity gradient "suppression" factor k, the more demanding the requirement that initial condition mismatches obey axis reversal in all drops of the n sets needed, which typically involve a long integration time [7].

The proposal [8] is therefore to be preferred. However, in real experiments exact compensation is not possible and a residual gravity gradient γ_{res} remains, for cloud A, or B or both, whereby:

$$\begin{aligned}\delta\Phi^A_{\Delta k} - \delta\Phi^B_{\Delta k} = \\ kT^2\Big\{(g^A_\circ - g^B_\circ) - \gamma_{res}\Big[(z^A_\circ - z^B_\circ) - (v^A_\circ - v^B_\circ)T\Big]\Big\} \ .\end{aligned} \quad (17)$$

The residual tidal acceleration

$$\Delta g_{tide} = \gamma_{res}[(z^A_\circ - z^B_\circ) - (v^A_\circ - v^B_\circ)T] \quad (18)$$

depending on the initial position and velocity offsets of the atom clouds, mimics a violation signal and needs to be separated from it for the WEP test to be meaningful.

A recent implementation in a dual ^{87}Rb and ^{85}Rb interferometer at the 10-m tower drop test of Stanford University has achieved a reduction by a factor of 100 [14]. The optimal value of Δk (in fact the corresponding laser frequency shift) which compensates the gravity gradient and minimizes the phase difference, is established empirically in a series of drops, from over-compensation to under-compensation, through best compensation (a procedure conceptually similar to that used with macroscopic bodies in order to achieve the best balancing or the best compensation of multiple mass moments' effects). In addition, the work reports a remarkable relative precision of $\Delta g/g \approx 6\times 10^{-11}$ per shot, while the previous best result, also with ^{87}Rb and ^{85}Rb, was at the 10^{-8} level [15]. For this experiment to achieve a WEP test to $\eta = 10^{-13}$ the number of drops needed is 3.6×10^5, and becomes 100 times larger for a 10^{-14} target.

To put things in context, rotating torsion balances have achieved $\eta \simeq 10^{-13}$ and this has been possible with a differential acceleration sensitivity which is not $10^{-13}\,g \simeq 10^{-12}\,\mathrm{ms}^{-2}$ as stated in [8], but $10^{-15}\,\mathrm{ms}^{-2}$ [16], because of the lower driving signal for test masses suspended on the surface of the Earth and rotating with it at the diurnal frequency [13]. Even the best test ever, based on preliminary data of the Microscope satellite in low Earth orbit, to 10^{-14} [17] has been obtained with a differential acceleration sensitivity about 70 times worse than torsion balances, exploiting the fact that the driving signal in orbit is the gravitational acceleration of the Earth at the orbiting altitude, similarly to mass dropping tests [18].

Atom interferometers are used also to perform differential measurements by means of spatially separated interferometers, with atom clouds of the same species, interrogated by the same laser beam (hence simultaneity is not a problem). While the physical quantity of interest, being the differential acceleration between different locations, is much smaller than the local acceleration itself, common mode effects are reduced. Gravity gradiometers based on atom interferometry are used for navigation and geodesy applications, but also for the measurement of the universal constant of gravity G and the detection of gravitational waves.

If the proposal [8] is applied exactly the gradiometer yields a measurement formally identical to (14) in which now $g_\circ^A - g_\circ^B$ is the differential acceleration measured by the gradiometer between location A and location B and the residual error is a factor $f = \frac{1}{12}\gamma T^2 \simeq 2.6 \times 10^{-7} T^2$ smaller than the differential acceleration measured by the gradiometer, as in (15). Whenever the differential acceleration of interest needs to be measured to this level, this systematic term must be taken into account and modelled.

As in the case of WEP tests, exact compensation is impossible. There will be a residual gravity gradient γ_{res} and a residual systematic tidal acceleration depending on the difference between the initial position and velocity of the two atom clouds in the different locations A and B of the gradiometer, as given by (18), which must be smaller than the target value of the physical observable of interest in all drops, and demonstrated to be so.

The proposal [8] has already been successfully applied to a cold atom gradiometer in order to improve the absolute measurement of G [19, 20]. We note that the $g_\circ T^2$ term with 7/12 coefficient is incorrectly missing from the starting Eq. (1) of [19], and that neglecting the residual term $\frac{1}{12} g_\circ T^2$ in Eq. (3) of both [19] and [20] is likely to be correct, but should nonetheless be justified in the absolute measurement of a fundamental physical constant.

We have shown that the acceleration measured by light-pulse AI systematically deviates from its true value by an acceleration term which is linear in the gravity gradient and quadratic in the time of fall. It does not affect tests of the UFF by atom interferometers, but is present when they are used as absolute gravimeters (for the measurement of g) or as gravity gradiometers (for the measurement of G and the detection of gravitational waves). The need to include this tidal term among systematic errors, and the extent to which it should be modelled, depends on the target of the experiment. It is already relevant for the absolute measurement of g.

A recent proposal to cancel the gravity gradient by applying an appropriate frequency shift at the second laser pulse overcomes the shortcomings of previous proposals and has been demonstrated to be effective. However, it does not eliminate the systematic deviation term outlined here. When applied to tests of the UFF, in which this term is negligible, it allows the requirements on the initial offsets between the atom species to be relaxed. The tests require that both species can be interrogated with the same laser pulses, and this is possible in the presence of a small isotope shift, as in the case of ^{87}Rb and ^{85}Rb.

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